Green's function - part II

Greens 2nd identity

$$\int_V d^3r' \left(\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2 \phi = -4\pi \rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r}, \vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$$

we saw one solution of above is $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$
but a more general solution is

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$$

where $\nabla'^2 F(\vec{r}, \vec{r}') = 0$, for $\vec{r}'$ in volume V
we will choose $F(\vec{r}, \vec{r}')$ to simplify solution of ϕ

$$\Rightarrow \int_V d^3r' \left(\phi(\vec{r}') \nabla'^2 G(\vec{r}, \vec{r}') - G(\vec{r}, \vec{r}') \nabla'^2 \phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left(\phi(r') \left[-4\pi \delta(\vec{r} - \vec{r}') \right] - G(\vec{r}, \vec{r}') \left[-4\pi \rho(\vec{r}') \right] \right)$$

$$= -4\pi \phi(\vec{r}) + 4\pi \int_V d^3r' \, G(\vec{r}, \vec{r}') \rho(\vec{r}')$$

$$= \oint_S da' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\Phi(\vec{r}) = \int_V d^3r' \, G(\vec{r},\vec{r}') \rho(\vec{r}') + \oint_S \frac{da'}{4\pi} \left(G(\vec{r},\vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(r') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} \right)$$

Consider <u>Dirichlet boundary problem</u>. If we can choose $F(\vec{r},\vec{r}')$ such that $G_D(\vec{r},\vec{r}')=0$ for $\vec{r}'$ on the boundary surface S, then above simplifies to

$$\boxed{\Phi(\vec{r}) = \int_V d^3r' \, G_D(\vec{r},\vec{r}') \rho(\vec{r}') - \oint_S \frac{da'}{4\pi} \phi(r') \frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}}$$

Since $\rho(r)$ is specified in V, and $\Phi(r)$ is specified on S, above then gives desired solution for $\Phi(r)$ inside volume V.

Finding G_D is therefore equivalent to finding an $F(r,r')$ such that $\nabla'^2 F(\vec{r},\vec{r}') \stackrel{?}{=} 0$ for $\vec{r}'$ in V (solves Laplace eqn) and

$F(\vec{r},\vec{r}') = \dfrac{-1}{|\vec{r}-\vec{r}'|}$ for $\vec{r}'$ on boundary surface S'

Always exists unique solution for F

Next consider Neumann boundary problem.

One might think to find $F(\vec{r},\vec{r}')$ such that $\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\int_V \nabla'^2 G(\vec{r},\vec{r}')\, d^3r' = \int \vec{\nabla}'\cdot\vec{\nabla}' G(\vec{r},\vec{r}')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}')\cdot\hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, da' = -4\pi$$

since $\nabla'^2 G = -4\pi\delta(r-r')$

So we can't have $\frac{\partial G}{\partial n'} = 0$ for $\vec{r}'$ on S

Simplest choice is then $\frac{\partial G_N(r,r')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}'$ on S

area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint \frac{da'}{4\pi}\, G(\vec{r},\vec{r}')\, \frac{\partial\phi(\vec{r}')}{\partial n'}$$

$$+ \oint \frac{da'}{4\pi}\, \phi(r') \left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}') = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}')\, \frac{\partial\phi(\vec{r}')}{\partial n'}\right]$$

$$+ \langle\phi\rangle_S$$

constant = average value of ϕ on surface S.

Since $\rho(r)$ is specified in V and $\frac{\partial\phi}{\partial n}$ is specified on S

above gives solution $\phi(r)$ in V within additive constant $\langle\phi\rangle_S$

Since $\vec{E} = -\vec{\nabla}\phi$ the const $\langle\phi\rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

and $$\frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While G_D and G_N always exist in principle, they depend in detail on the shape of the surface S and are difficult to find except for simple geometries

In preceeding we defined G by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface S', $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of G above is obtained by noting that one can prove the symmetry property

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10)