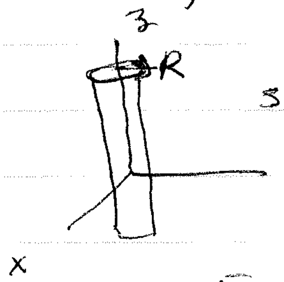
Some more problems

infinite conducting wire of radius R with line charge density λ = charge per unit length



surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord r.

$\nabla^2 \phi = 0$ for $r > R$, $r < R$

use ∇^2 in cylindrical coords – only radial term non vanishing

$$\nabla^2 \phi = \frac{1}{r} \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) = 0$$

$$r \frac{d\phi}{dr} = C_0 \quad \text{constant}$$

$$\frac{d\phi}{dr} = \frac{C_0}{r}$$

$$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$$

note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

one needs to fix zero of ϕ at some other radius. a convenient choice is $r = R$, but any other choice could also be made.

$$\phi^{out} = C_0^{out} \ln r + C_1^{out}$$
$$\phi^{in} = C_0^{in} \ln r + C_1^{in}$$

ϕ^{in} = const in conductor $\Rightarrow C_0^{in} = 0$

or ϕ^{in} should not diverge as $r \to 0 \Rightarrow C_0^{in} = 0$

So $\phi^{in} = C_1^{in}$ constant

boundary condition at $r = R$

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

$$\Rightarrow \quad -\frac{C_0^{out}}{R} = 4\pi\sigma = 4\pi\left(\frac{\lambda}{2\pi R}\right) = \frac{2\lambda}{R}$$

$$C_0^{out} = -2\lambda$$

$$\phi^{out}(r) = -2\lambda \ln r + C_1^{out}$$

continuity of ϕ

$$\phi^{in}(R) = \phi^{out}(R) \quad \Rightarrow \quad C_1^{in} = -2\lambda \ln R + C_1^{out}$$

Remaining const C_1^{out} is not too important as it is just a common additive constant to both ϕ^{in} and ϕ^{out} $\Rightarrow$ does not change $\vec{E} = -\vec{\nabla}\phi$.

If use the condition $\phi(R) = 0$ then we can solve for C_1^{out}.

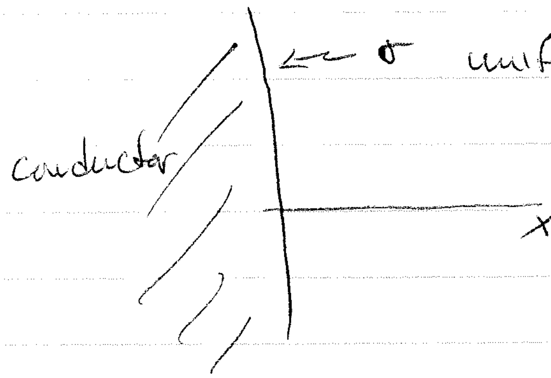
$$0 = -2\lambda \ln R + c_1^{out} \Rightarrow c_1^{out} = 2\lambda \ln R$$

$$\Rightarrow \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{2\lambda}{r}\hat{r} & r > R \\ 0 & r < R \end{cases}$$

infinite conducting half space

$\leftarrow \sigma$ uniform surface charge density



expect ϕ depends only on x

$$\nabla^2 \phi = \frac{d^2\phi}{dx^2} = 0$$

$$\Rightarrow \begin{cases} \phi^>(x) = c_0^> x + c_1^> & x > 0 \\ \phi^<(x) = c_0^< x + c_1^< & x < 0 \end{cases}$$

for $x < 0$, $\phi = $ const in conductor $\Rightarrow c_0^< = 0$

at $x = 0$, ϕ continuous $\Rightarrow \phi^<(0) = \phi^>(0)$

$$c_1^< = c_1^>$$

$\frac{d\phi}{dx}$ discontinuous $\Rightarrow$

$$-\left.\frac{d\phi^>}{dx}\right|_{x=0} = 4\pi\sigma$$

$$c_0^> = -4\pi\sigma$$

$$\Rightarrow \phi(x) = \begin{cases} -4\pi\sigma x + c_1^> & x > 0 \\ c_1^> & x < 0 \end{cases}$$

const $c_1^>$ does not change value of $\vec{E}$

as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$.
~~we can set $\phi = 0$ at~~

$$-\vec{\nabla}\phi = \vec{E} = \begin{cases} 4\pi\sigma\,\hat{x} & x > 0 \\ 0 & x < 0 \end{cases}$$

<u>infinite charged plane</u>

similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

~~expect the dependence only on $|x|$ by symmetry~~

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \begin{aligned} \phi^> &= c_0^> x + c_1^> & x > 0 \\ \phi^< &= c_0^< x + c_1^< & x < 0 \end{aligned}$$

continuity of ϕ at $x = 0$

$$\to \quad \phi^>(0) = \phi^<(0) \quad \Rightarrow \quad c_1^> = c_1^<$$

discontinuity of $d\phi/dx$ at $x = 0$

$$-\frac{d\phi^>}{dx} + \frac{d\phi^<}{dx} = 4\pi\sigma$$

$$-c_0^> + c_0^< = 4\pi\sigma$$

Define $\bar{c}_0 = \dfrac{c_0^> + c_0^<}{2}$

Then we can write

$$C_0^< = \bar{C}_0 + 2\pi\sigma$$
$$C_0^> = \bar{C}_0 - 2\pi\sigma$$

$$\phi = \begin{cases} -2\pi\sigma x + \bar{C}_0 x + C_1^> & x > 0 \\ 2\pi\sigma x + \bar{C}_0 x + C_1^> & x < 0 \end{cases}$$

$$\Rightarrow \quad -\frac{d\phi}{dx} = \vec{E} = \begin{cases} (2\pi\sigma - \bar{C}_0)\,\hat{x} & x > 0 \\ (-2\pi\sigma - \bar{C}_0)\,\hat{x} & x < 0 \end{cases}$$

Const $C_1^>$ does not effect $\vec{E}$ - additive const to ϕ
$\bar{C}_0$ represents const uniform electric field $-\bar{C}_0\hat{x}$, that exists independently of the charged surface

If we assumed that all $\vec{E}$ fields are just those arising from the plane, then we can set $\bar{C}_0 = 0$.
Equivalently, if the plane is the only source of $\vec{E}$, then we expect ϕ depends only on $|x|$ by symmetry.
$\Rightarrow \quad C_0^< = -C_0^>$ and again $\bar{C}_0 = 0$. In this case

$$\phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases}$$

(we also set $C_1^> = 0$ here corresponding to $\phi(0) = 0$

$$\vec{E}(x) = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$

$\vec{E}$ is constant ~~by~~ but oppositely directed on either side of the charged plane

Green's Theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - ie there is a unique solution. We start by deriving <u>Greens Theorems</u>

Consider $\int_V d^3r\, \vec\nabla\cdot\vec A = \oint_S da\, \hat m\cdot\vec A$ Gauss theorem

let $\vec A = \phi\vec\nabla\psi$ ϕ,ψ any two scalar functions

$$\Rightarrow \quad \vec\nabla\cdot\vec A = \phi\nabla^2\psi + \vec\nabla\phi\cdot\vec\nabla\psi$$

$$\phi\,\vec\nabla\psi\cdot\hat m = \phi\frac{\partial\psi}{\partial m}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec\nabla\phi\cdot\vec\nabla\psi\right) = \oint_S da\,\phi\frac{\partial\psi}{\partial m} \quad \Big\}\ \text{Green's 1st-identity}$$

let $\phi\leftrightarrow\psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec\nabla\phi\cdot\vec\nabla\psi\right) = \oint_S da\,\psi\frac{\partial\phi}{\partial m}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial m} - \psi\frac{\partial\phi}{\partial m}\right) \quad \Big\}\ \text{Green's 2nd identity}$$

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec r-\vec r'|}$, $\vec r'$ is integration variable, ϕ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. Use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec r-\vec r')$

$$\int_V d^3r'\left[\phi(r')\left[-4\pi\,\delta(r-r')\right] - \left(\frac{1}{|\vec r-\vec r'|}\right)\left(-4\pi\rho(\vec r')\right)\right]$$

$$= \oint_S da'\left[\phi\frac{\partial}{\partial m'}\left(\frac{1}{|\vec r-\vec r'|}\right) - \frac{1}{|\vec r-\vec r'|}\frac{\partial\phi}{\partial m'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

Note: if $\vec{r}$ lies outside the volume V, then

$$(**) \qquad 0 = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

potential from a surface charge density $\sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}$

potential from a surface dipole layer of dipole strength density $\frac{\phi}{4\pi}$

From (*), if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int_V d^3r' \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

(*) gives the generalization of Coulombs law to a system with a finite boundary

For a <u>charge free</u> volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify <u>both</u> ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting ϕ from (*) would not in general obey Laplace's equation $\nabla^2\phi = 0$, nor would (**) vanish.

Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

Uniqueness

If we have a system of charges in vol V, and *either* the potential ϕ, *or* its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions ϕ_1 and ϕ_2, *both* with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S – for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface S – for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \, (\underbrace{u\nabla^2 u}_{\substack{\shortparallel \\ 0 \\ \text{as } \nabla^2 u = 0}} + \vec{\nabla} u \cdot \vec{\nabla} u) = \oint_S da \, \underbrace{u \frac{\partial u}{\partial n}}_{\substack{\shortparallel \\ 0 \\ \text{as } u \text{ or } \frac{\partial u}{\partial n} = 0}}$$

$\Rightarrow \int_V d^3r \, |\vec{\nabla} U|^2 = 0 \qquad \Rightarrow \vec{\nabla} U = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow U = \text{const}$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

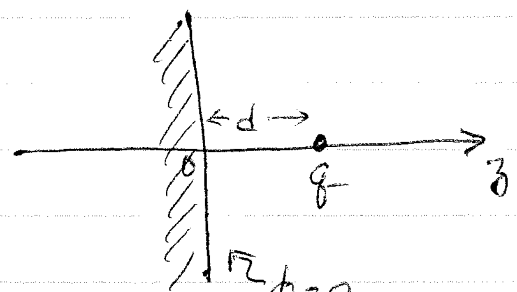
~~[illegible]~~ If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial\phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial\phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial\phi}{\partial n}$ alone is enough to give a unique solution.

Image Charge method

For simple geometries, can try to obtain G_D or G_N by placing a set of "image charges" outside the volume of interest V, ie on the "other side" of the system boundary surface S. Because these image charges are outside V, their contrib to the potential inside V obeys $\nabla^2\phi^{\text{image}} = 0$, as necessary. Choose location of image charges so that total ϕ has desired boundary condition.

1) charge in front of infinite grounded plane



want $\nabla^2\phi = -4\pi q\,\delta(x)\delta(y)\delta(z-d)$

$\phi = 0$ for $z=0$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z=-d$

ϕ is Coulomb potential from the real charge + the image

$$\phi(\vec{r}) = \underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{real charge}} + \underbrace{\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge}}$$

above satisfies $\phi(x,y,0) = 0$ as required

Also, $\nabla^2\phi = -4\pi q\,\delta(\vec{r}-d\hat{z}) + 4\pi q\,\delta(\vec{r}+d\hat{z})$

$= -4\pi q\,\delta(\vec{r}-d\hat{z})$ for region $z>0$