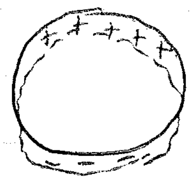
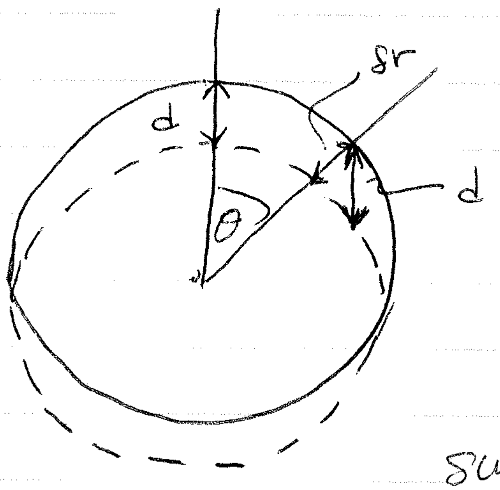
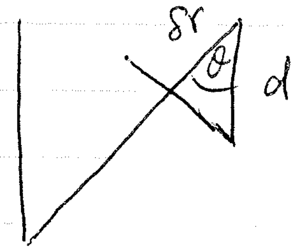
Physical exaple with $\sigma(\theta) = k\cos\theta$

Two spheres of radii R, with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement
This is a uniformly "polarized" sphere





$d\cos\theta = \delta r$

surface charge is $\sigma(\theta) = \rho\,\delta r$
$= \rho d \cos\theta$

$$\boxed{\sigma(\theta) = \rho d \cos\theta}$$

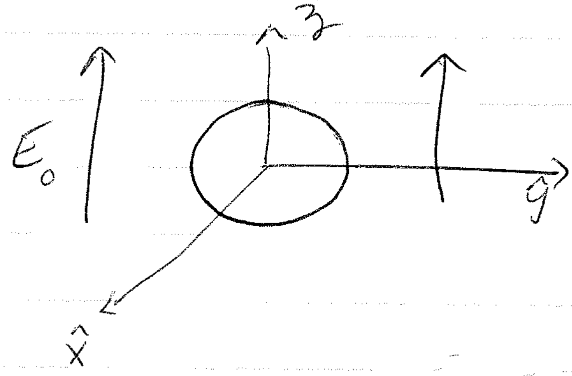
total dipole moment is $(\rho d)\frac{4}{3}\pi R^3$

polarization $= \dfrac{\text{dipole moment}}{\text{volume}} = \rho d$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}$

Grounded
(3) Conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$
$= -E_0 r \cos\theta$



boundary conditions

$$\begin{cases} \phi(R,\theta) = 0 \\ \phi(r \to \infty, \theta) = -E_0 r \cos\theta \end{cases}$$

solution outside sphere has the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta)$$

From boundary condition as $r \to \infty$ we have
$A_l = 0$ all $l \neq 1$
$A_1 = -E_0$ since $P_1(\cos\theta) = \cos\theta$

$$\phi(r,\theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

From $\phi(R,\theta) = 0$ we have

$$0 = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$$\Rightarrow B_l = 0 \text{ all } l \neq 1$$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = +E_0 R^3$$

So $$\boxed{\phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

1st term is just potential $-E_0 r\cos\theta$ of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the surface - it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\frac{\partial\phi}{\partial r}\bigg|_{r=R} = E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta$$

$$= 3E_0\cos\theta$$

$$\sigma(\theta) = \frac{3}{4\pi}E_0\cos\theta$$ like uniformly polarized sphere $k = \frac{3E_0}{4\pi}$

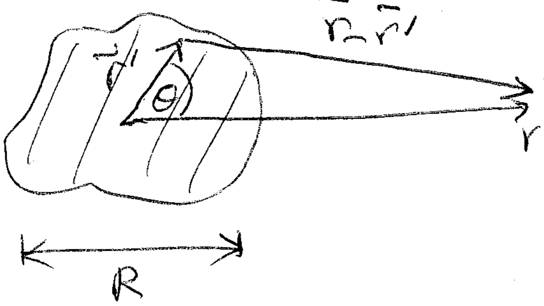
from ② we know that the field inside the sphere due to this σ is just $-\frac{4}{3}\pi k\hat{z} = -\frac{4}{3}\pi\frac{3E_0}{4\pi}\hat{z}$

$= -E_0\hat{z}$. This is just what is required so that the total field in the conducting sphere vanishes.

Can check that outside the sphere, $\vec{E} = -\vec{\nabla}\phi$ is normal to surface of sphere at $r=R$.

Multipole Expansion

region with $\rho \neq 0$

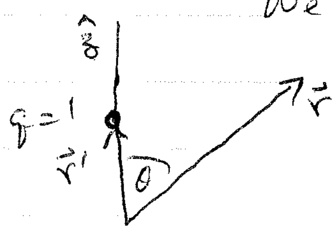


We want to find the potential ϕ for an arbitrary <u>localized</u> distribution of charge ρ, at distances far away $r >> R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ General Coulomb formula

We want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r >> r'$

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ, so we can express it as an expansion in Legendre polynomials.

<u>For $r > r'$,</u>

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

all $A_\ell = 0$ as need $\phi \to 0$ as $r \to \infty$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} P_\ell(\cos\theta)$$

We have $\phi(r, \theta=0) = \dfrac{1}{r-r'}$ (for $r > r'$)

$\uparrow$ scalars here since when $\theta = 0$ $\vec{r}$ and $\vec{r}'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r,0) = \frac{1}{r}\sum_\ell \frac{B_\ell}{r^\ell} P_\ell(1)$$

$$= \frac{1}{r}\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} \quad \text{as } P_\ell(1) = 1$$

$$= \frac{1}{r}\frac{1}{(1 - r'/r)} \leftarrow \text{exact result from Coulomb}$$

Now Taylor expansion $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$

$$\Rightarrow \frac{1}{r}\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots\right)$$

$\Rightarrow \quad B_\ell = (r')^\ell$ is solution

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)}$$

So for the charge distribution ρ,

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \int d^3r' \frac{\rho(\vec{r}')}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \rho(\vec{r}')(r')^\ell P_\ell(\cos\theta)$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r). It is exact provided one adds all the infinite l terms. In practice, one generally approximates by summing only up to some finite l.

Note: in doing the integrals

$$\int d^3r' \, \rho(\vec{r}\,')(r')^l P_l(\cos\theta)$$

θ is defined as the angle of $\vec{r}\,'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat this integration every time we change $\vec{r}$.

We will find a way around this by
(i) first looking explicitly at the few lowest order terms
(ii) a general method involving spherical harmonics $Y_{lm}(\theta, \varphi)$

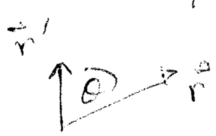
monopole: $l=0$ term

$$\phi^{(0)}(\vec{r}) = \frac{1}{r}\int d^3r'\,\rho(r') \qquad P_0(\cos\theta) = 1$$

$$= \frac{q}{r}$$ where $q \equiv \int d^3r'\,\rho(r')$ is total charge

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}')\,r'\,P_1(\cos\theta)$$



$$= \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}')\,r'\cos\theta$$

Now $\vec{r}\cdot\vec{r}' = rr'\cos\theta \Rightarrow \hat{r}\cdot\vec{r}' = r'\cos\theta$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\,\hat{r}\cdot\int d^3r'\,\rho(\vec{r}')\,\vec{r}'$$

$$= \frac{\vec{p}\cdot\hat{r}}{r^2}$$ where $\vec{p} \equiv \int d^3r'\,\rho(\vec{r}')\,\vec{r}'$ is the dipole moment

For a set of point charges q_i at $\vec{r}_i$,
$$\vec{p} = \sum_i q_i\vec{r}_i$$

quadrapole: $\ell = 2$ term

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, r'^2 P_2(\cos\theta)$$

$$= \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, r'^2\, \tfrac{1}{2}(3\cos^2\theta - 1)$$

use $\cos\theta = \hat{r}'\cdot\hat{r}$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\, \rho(\vec{r}')\, \tfrac{1}{2}\left(3(\vec{r}'\cdot\hat{r})^2 - (r')^2\right)$$

$$= \frac{1}{r^3}\, \hat{r}\cdot\left[\int d^3r'\, \rho(\vec{r}')\, \tfrac{1}{2}\left(3\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)\right]\cdot\hat{r}$$

where $\overleftrightarrow{I}$ is the identity tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot\overleftrightarrow{I}\cdot\vec{v} = \vec{u}\cdot\vec{v}$.
and $\vec{r}'\vec{r}'$ is the tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot[\vec{r}'\vec{r}']\cdot\vec{v} = (\vec{u}\cdot\vec{r}')(\vec{r}'\cdot\vec{v})$

Define quadrapole tensor $\overleftrightarrow{Q} \equiv \int d^3r'\, \rho(\vec{r}')\, \left(3\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\,\frac{1}{2}\,\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}$$

So to lowest three terms

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} + \cdots$$

defined in terms of the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ of the charge distribution.

Note, the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ do not depend on the observation point $\vec{r}$ — we can calculate them once and then use them to get $\phi(\vec{r})$ at all $\vec{r}$.

monopole: $\quad q = \int d^3r\, \rho(\vec{r})$ $\quad$ scalar integral

dipole $\quad \vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r}$ $\quad$ vector integral

$\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

if we pick a coordinate system, we have to do $\underline{\underline{3}}$ integrations to get the three component of $\vec{p}$

$$\hat{e}_i \cdot \vec{p} = p_i = \int d^3r\, \rho(\vec{r})\, r_i$$

quadrapole $\quad \overleftrightarrow{Q} = \int d^3r\, \rho(\vec{r}) \left(3\vec{r}\,\vec{r} - (r)^2 \overleftrightarrow{I}\right)$ $\quad$ tensor integral

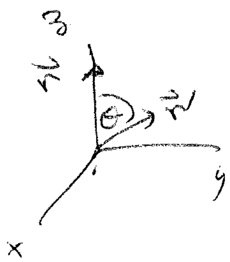
if we pick a coord system $x\ y\ z$ then $\overleftrightarrow{Q}$ is a matrix with components $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

$$\hat{e}_i \cdot \overleftrightarrow{Q} \cdot \hat{e}_j = Q_{ij} = \int d^3r\, \rho(\vec{r}) \left[3 r_i r_j - r^2 \delta_{ij}\right]$$

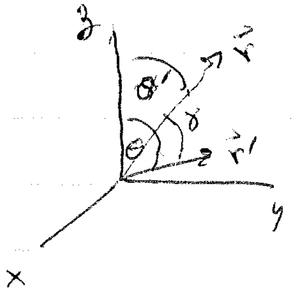
There are 9 elements of the 3×3 matrix Q_{ij}, but $Q_{ij} = Q_{ji}$ is symmetric so there are only 6 independent elements to compute.

General method

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d^3r' \, \rho(\vec{r}')(r')^l \, P_l(\cos\theta)$$



in above, θ is angle between $\vec{r}$ and $\vec{r}'$
if we think of this θ as the spherical coord θ, then in effect, above is choosing $\vec{r}$ to be on $\hat{z}$ axis. We would like a representation in which $\vec{r}$ is positioned arbitrarily with respect to the axes used in describing ρ



Use the <u>addition theorem for spherical harmonics</u> - see Jackson 3.6 for discussion & proof

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi') \, Y_{lm}(\theta, \phi)$$

where (θ, ϕ) are the angles of $\hat{r}$, (θ', ϕ') are the angles of $\hat{r}'$, and γ is the angle between $\hat{r}$ and $\hat{r}'$, ie $\cos\gamma = \hat{r}\cdot\hat{r}'$

$\cos\theta = \hat{z}\cdot\hat{r}$

$\cos\theta' = \hat{z}\cdot\hat{r}'$

$\Rightarrow$

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \int d^3r' \, \rho(\vec{r}')(r')^l \, Y^*_{lm}(\theta', \phi') \, Y_{lm}(\theta, \phi)$$

Define the moment

$$q_{lm} \equiv \int d^3r' \, \rho(\vec{r}')(r')^l \, Y^*_{lm}(\theta', \phi')$$

independent of observation point

Then

$$\phi(\vec{r}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell m} Y_{\ell m}(\theta,\phi)}{(2\ell+1)\, r^{\ell+1}}$$

see Jackson eqn (4.4), (4.5), (4.6) to relate $q_{\ell m}$ to q, $\vec{p}$, $\overleftrightarrow{Q}$.

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} \cdots$$

electric field $\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$

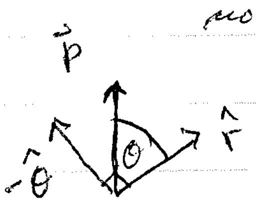
For the monopole term $\vec{E} = \frac{q}{r^2}\hat{r}$

For the dipole term, choose $\vec{p}$ along $\hat{z}$ axis so

$$\phi(\vec{r}) = \frac{p\cos\theta}{r^2}$$

$$\vec{E} = \frac{2p\cos\theta}{r^3}\hat{r} + \frac{p\sin\theta}{r^3}\hat{\theta}$$

$$\vec{E} = \frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



note $\quad p\cos\theta\,\hat{r} = (\vec{p}\cdot\hat{r})\hat{r}$

$\qquad\quad p\sin\theta\,\hat{\theta} = -(\vec{p}\cdot\hat{\theta})\hat{\theta}$

Now $\quad \vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$

$\Rightarrow \quad -(\vec{p}\cdot\hat{\theta})\hat{\theta} = (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$

So

$$\vec{E} = \frac{1}{r^3}\left[2(\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

$$= \frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$ expresses $\vec{E}$ in coord free form