$$\vec{E} = \frac{1}{r^3}\left[\, 3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\,\right]$$

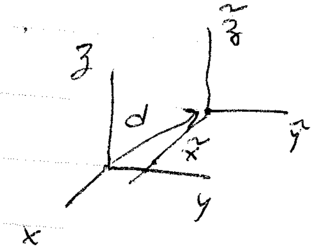
expresses $\vec{E}$ of dipole in coord free form

<u>Origin of coordinates</u>

The definition of the multipole moments <u>depends</u> on the choice of origin of the coordinates

Suppose transform to $\tilde{\vec{r}} = \vec{r} - \vec{d}$

In the $\tilde{\vec{r}}$ cood system



$$\tilde{q} = \int d^3\tilde{r}\ \rho(\tilde{r}) = \int d^3r\ \rho(r) = q$$

monopole <u>does not</u> depend on choice of origin

$$\tilde{\vec{p}} = \int d^3\tilde{r}\ \rho(\tilde{r})\,\tilde{\vec{r}} = \int d^3r\ \rho\,(\vec{r} - \vec{d}\,)$$

$$= \int d^3r\ \rho\,\vec{r} - \vec{d}\int d^3r\,\rho$$

$$\tilde{\vec{p}} = \vec{p} - \vec{d}q \qquad \tilde{\vec{p}} = \vec{p} \text{ only if } q = 0\,!$$

if $q \neq 0$, then $\tilde{\vec{p}} \neq \vec{p}$

$\Rightarrow$ ~~One coord~~ If $q \neq 0$, one could always choose an origin of coords for which $\vec{p} = 0$!

~~For HW you will show that~~ $\tilde{\vec{p}} - \ldots$

Quadrupole moment in new coordinates

$$\overleftrightarrow{\tilde{Q}} = \int d^3\tilde{r}\, \rho \left[3\tilde{\vec{r}}\,\tilde{\vec{r}} - (\tilde{\vec{r}})^2 \overleftrightarrow{I} \right]$$

where $\tilde{\vec{r}} = \vec{r} - \vec{d}$

substitute in above

$$\overleftrightarrow{\tilde{Q}} = \int d^3r\, \rho \left[3(\vec{r}-\vec{d})(\vec{r}-\vec{d}) - (\vec{r}-\vec{d})^2 \overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - 3\vec{r}\,\vec{d} - 3\vec{d}\,\vec{r} + 3\vec{d}\,\vec{d} - (r^2 + d^2 - 2\vec{r}\cdot\vec{d})\overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - r^2 \overleftrightarrow{I} \right] - 3\left[\int d^3r\, \rho\, \vec{r} \right] \vec{d} - 3\vec{d} \left[\int d^3r\, \rho\, \vec{r} \right]$$

$$+ 3\vec{d}\,\vec{d} \left[\int d^3r\, \rho \right] - d^2 \overleftrightarrow{I} \left[\int d^3r\, \rho \right]$$

$$+ 2\left[\int d^3r\, \rho\, \vec{r} \right] \cdot \vec{d}\, \overleftrightarrow{I}$$

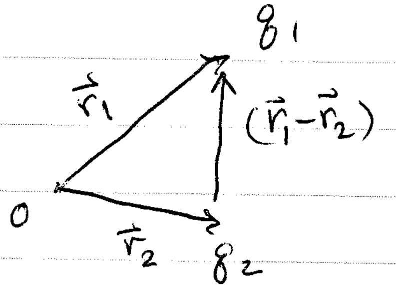
$$\overleftrightarrow{\tilde{Q}} = \overleftrightarrow{Q} - 3\vec{p}\,\vec{d} - 3\vec{d}\,\vec{p} + 3\vec{d}\,\vec{d}\, q - \left(d^2 q - 2\vec{p}\cdot\vec{d} \right) \overleftrightarrow{I}$$

we see that $\overleftrightarrow{Q}$ is independent of choice of origin only when both q and $\vec{p}$ vanish. When this happens the quadrupole term is the <u>leading</u> term in the multipole expansion.

In general, the leading term in multipole expansion will be indep of origin of coordinates.

Example   two charges q_1 at $\vec{r}_1$ and q_2 at $\vec{r}_2$

$$q_1 + q_2 = q \neq 0$$



monopole $\quad q_1 + q_2 = q$

dipole $\quad \vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $\quad \overleftrightarrow{Q} = (3\vec{r}_1\vec{r}_1 - r_1^2 \overleftrightarrow{I})q_1 + (3\vec{r}_2\vec{r}_2 - r_2^2 \overleftrightarrow{I})q_2$

We can make the dipole moment vanish by shifting to a new coord system $\vec{r}\,' = \vec{r} - \vec{d}$ where $\vec{d} = \dfrac{\vec{p}}{q}$

$$\vec{r}\,' = \vec{r} - \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2} = \frac{q_1(\vec{r} - \vec{r}_1) + q_2(\vec{r} - \vec{r}_2)}{q_1 + q_2}$$

positions of q_1, q_2 in new coords are

$$\vec{r}_1\,' = \frac{q_2}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

Origin of new coord system is at

lies along vector from $\vec{r}_2$ to $\vec{r}_1$

$$\vec{r}\,' = 0 \Rightarrow \vec{r} = \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2}$$ "center of charge"

for many charges q_i at positions $\vec{r}_i$, the origin that makes dipole moment vanish is at

$$\vec{r} = \frac{\sum_i q_i \vec{r}_i}{\sum q_i}$$

In this coord system

$$\vec{p}\,' = q_1 \vec{r}_1\,' + q_2 \vec{r}_2\,' = \frac{q_1 q_2}{q_1+q_2}(\vec{r}_1 - \vec{r}_2) - \frac{q_2 q_1}{q_1+q_2}(\vec{r}_1 - \vec{r}_2)$$

$= 0$ as it must be!

<u>Quadrupole moment</u> in the coord system in which $\vec{p}\,' = 0$ the quadrupole tensor is

$$\overleftrightarrow{Q}\,' = \left[3\vec{r}_1\,'\vec{r}_1\,' - (r_1')^2 \overleftrightarrow{I}\right] q_1 + \left[3\vec{r}_2\,'\vec{r}_2\,' - (r_2')^2 \overleftrightarrow{I}\right] q_2$$

let us choose ~~coord~~ spherical coordinates with origin at O' and $\hat{z}$ axis aligned along $\vec{r}_1 - \vec{r}_2$, so that

$\vec{r}_1 - \vec{r}_2 = s\,\hat{z}$ where $s = |\vec{r}_1 - \vec{r}_2|$ is separation between the charges

Then $\vec{r}_1\,' = \dfrac{q_2}{q_1+q_2}\, s\,\hat{z}$

$\vec{r}_2\,' = \dfrac{-q_1}{q_1+q_2}\, s\,\hat{z}$

$$\overleftrightarrow{Q}\,' = \left(\frac{q_2}{q_1+q_2}\right)^2 q_1 \left[3 s^2 \hat{z}\hat{z} - s^2 \overleftrightarrow{I}\right] + \left(\frac{-q_1}{q_1+q_2}\right)^2 q_2 \left[3 s^2 \hat{z}\hat{z} - s^2 \overleftrightarrow{I}\right]$$

$$\overleftrightarrow{Q}' = \frac{q_2^2 q_1 + q_1^2 q_2}{(q_1+q_2)^2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$= \frac{q_1 q_2}{q_1+q_2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$Q'_{ij} = \frac{q_1 q_2}{q_1+q_2} s^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in xyz coord system

as $\hat{z}\hat{z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

the contribution of quadrupole to the potential is

$$\Phi_{quad} = \frac{1}{2} \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{r^3}$$

$\hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$ in xyz coords

with origin at O' this becomes

$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

do matrix multiplications

$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (2\cos^2\theta - \sin^2\theta)$$

independent of φ as it must be due to azimuthal symmetry

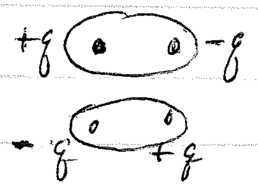
Example

sample charge config's

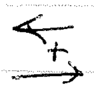
$\circ\ q$ $\Rightarrow$ monopole is leading term

$\overset{\circ}{+q}$ $\overset{\circ}{-q}$ $\Rightarrow$ monopole $= 0$ $\Rightarrow$ dipole is leading term

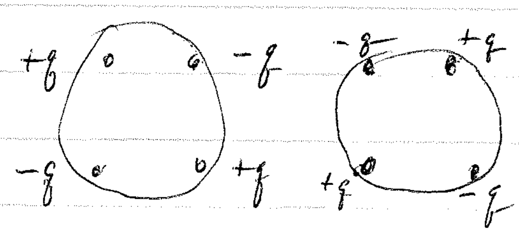
$\vec{p}$ is indep of origin



$\Rightarrow$ monopole $= 0$ $\Rightarrow$ total dipole is sum of dipoles of individual neutral pairs



$= 0$ leading term is quadrapole



when monopole $= 0$ and dipole $= 0$, quadrapole is indep of origin.

$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q = 0$ and $\vec{p} = 0$

$Q = Q_1 + Q_2$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is <u>octopole</u>

Magnetostatics

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{j} \end{cases}$$ Ampere's Law (statics only!)

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a <u>simple</u> expression in <u>Cartesian coords</u>

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2\vec{A} = \nabla^2\left(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}\right)$$

$$\cancel{= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta})}$$
$$\cancel{+ (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}}$$ Poisson's equation!

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of equ makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $\vec{j}(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r} \frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \underbrace{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}}_{①} + \underbrace{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r}\cdot\vec{r}')}{r^3}}_{②} + \ldots$$

<u>Consider ~~term~~ ①</u>

$$\int d^3r\, \vec{j}(\vec{r}) \qquad \int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\displaystyle \int d^3r\, j_i(r) = \sum_{j=1}^{3} \int d^3r\, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left\{ \oint_S da\, j_j r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r\, \vec{j}(\vec{r}) = 0$ in magnetostatics
monopole term vanishes

term ②

$$\int d^3r \; \vec{j}(\vec{r}) \, \vec{r} \qquad \text{tensor}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k} (j_k r_j) r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= - \sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla} \cdot \vec{j} = 0$ in magnetostatics; $\frac{\partial r_j}{\partial r_k} = \delta_{jk}$

$$= - \int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = - \int d^3r \; j_j r_i$

$$= \frac{1}{2} \int d^3r \left(j_i r_j - j_j r_i \right)$$

So

$$\int d^3r' \; j_i(\vec{r}')(\vec{r} \cdot \vec{r}') = \sum_j r_j \int d^3r' \, j_i(\vec{r}') \, r_j'$$

$$= \sum_j \frac{1}{2} \int d^3r' \left(j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2} \int d^3r' \left(j_i (\vec{r} \cdot \vec{r}') - r_i' (\vec{r} \cdot \vec{j}) \right)$$