use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{j}) = \vec{r}'(\vec{r} \cdot \vec{j}) - \vec{j}(\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \, \vec{j} \, (\vec{r} \cdot \vec{r}') = -\frac{1}{2} \, \vec{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}') \right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}')}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

$$B_i = \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, m_l \, \frac{r_m}{r^3}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \, \partial_j \, m_l \, \frac{r_m}{r^3}$$

$$= m_i \, \partial_j \left(\frac{r_j}{r^3} \right) - m_j \, \partial_j \left(\frac{r_i}{r^3} \right)$$

$$= m_i \left[-4\pi \delta(\vec{r}) \right] - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i}{r^4} \, \partial_j r \right]$$

$$= 0 - \frac{m_i}{r^3} + 3 \frac{r_i}{r^4} \frac{r_j}{r} m_j$$

far from source

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{any two indices equal} \\ +1, & ijk \text{ even permutat. of } 123 \\ -1, & ijk \text{ odd permutat. } 123 \end{cases}$$

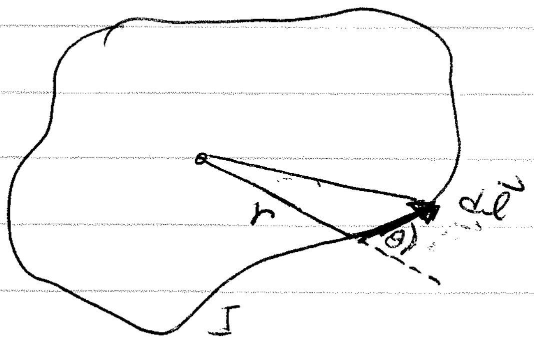
$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

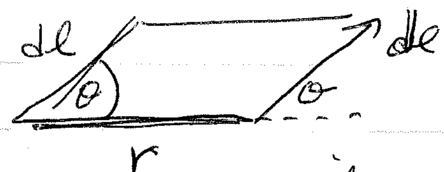
same form as $\vec{E}$ from electric dipole $\vec{p}$

For a <u>current loop in a plane</u> (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$
$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of ~~rectangle~~ trapazoid is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I (\text{area})\, \hat{n}$$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r'\,(\vec{r}\,'\times\vec{j}\,) = \frac{1}{2c}\int d^3r\;(\vec{r}+\vec{d}\,)\times\vec{j}$$

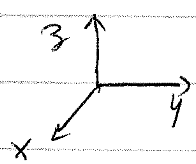
$$= \frac{1}{2c}\int d^3r\;\vec{r}\times\vec{j} + \frac{1}{2c}\vec{d}\times\left[\int d^3r\;\vec{j}\,\right]$$

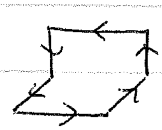
$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r\;\vec{j} = 0$$

---

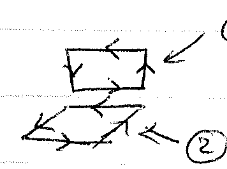
for planar loop $\vec{m} = \frac{I}{c}a\,\hat{n}$ where a = area, $\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>





= superposition of



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c}a_1\,\hat{x}$$

$$\vec{m}_2 = \frac{I}{c}a_2\,\hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}\left(a_1\hat{x} + a_2\hat{z}\right)$$

Boundary value problems in magnetostatics
Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2 \phi = 4\pi \rho$ dont work so well for magnetostatics.

However, in situations where the current $\vec{J}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M.

In regions where $\vec{J} = 0$, ie not on the certain surfaces, we have $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Since $\vec{\nabla} \times \vec{B} = 0$ in these regions we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

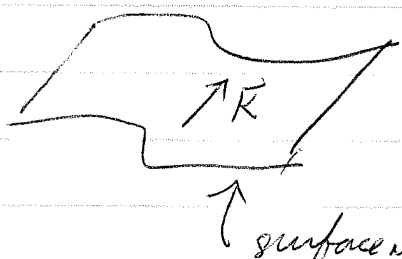
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Boundary Conditions at Sheet current

in magnetostatics $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$



surface current $\vec{K}(\vec{r})$ at pt $\vec{r}$ on surface S

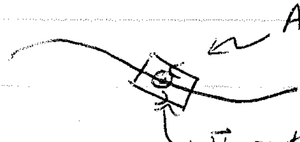
Gaussian pillbox vol V $\quad \int_V d^3r\, \vec{\nabla}\cdot\vec{B} = 0$

side view

top + bottom area of pill box is da
width of pill box $\to 0$

$$\Rightarrow \int_V d^3r\, \vec{\nabla}\cdot\vec{B} = \oint_S da\, \hat{n}\cdot\vec{B} = da\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\hat{n} = 0$$

normal component of $\vec{B}$ is continuous $\quad (\vec{B}_{above} - \vec{B}_{below})\cdot\hat{n} = 0$



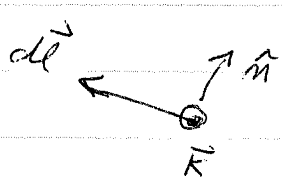
Amperian loop C bounding area S

$\vec{K}$ out of page

side view

$$\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\to 0$, top + bottom sides $d\vec{l}$



$$(\vec{B}_{above} - \vec{B}_{below})\cdot d\vec{l} = \frac{4\pi}{c}(\hat{n}\times d\vec{l})\cdot\vec{K}$$
$$= \frac{4\pi}{c}(\vec{K}\times\hat{n})\cdot d\vec{l}$$

$\hat{n}$ is outward normal

tangential component of $\vec{B}$ has discontinuous jump $\frac{4\pi}{c}\vec{K}\times\hat{n}$

Combine both results into

$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c} \vec{K} \times \hat{n}}$$

magnetic analog of $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~ scalar potential ϕ_M

$$-\vec{\nabla}\phi_{M\,above} + \vec{\nabla}\phi_{M\,below} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ.
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q\left[\phi(r_2) - \phi(r_1)\right]$
nothing similar for ϕ_M.
(in fact ~~magnetostatic~~ magnetic forces do no work!
$\vec{F} = q\,\vec{v} \times \vec{B}$
$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0$)

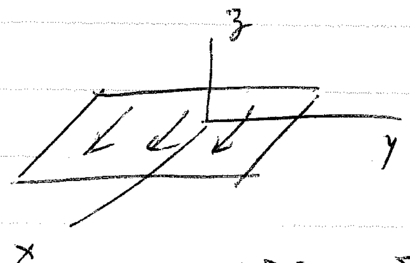
Note: ~~We cannot apply argument like $\phi(r) - \phi(r') = -\int_{r'}^{r} \vec{E} \cdot d\vec{l}$~~
ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use

$$\phi_M(r_{above}) - \phi_M(r_{below}) = -\int_{r_{below}}^{r_{above}} \vec{B} \cdot d\vec{l}$$

since ϕ_M is not defined on the current sheet
itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current
$\vec{K} = K\hat{x}$



$$z>0, \quad \nabla^2 \Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z<0, \quad \nabla^2 \Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z>0, \quad \vec{B}^> = -\vec{\nabla}\Phi_M^> = b_x^> \hat{x} + b_y^> \hat{y} + b_z^> \hat{z}$$
$$z<0, \quad \vec{B}^< = -\vec{\nabla}\Phi_M^< = b_x^< \hat{x} + b_y^< \hat{y} + b_z^< \hat{z}$$

at $z=0$ $\quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<)\hat{x} + (b_y^> - b_y^<)\hat{y} + (b_z^> - b_z^<)\hat{z}$
$$= \frac{4\pi}{c}\vec{K}\times\hat{n} = \frac{4\pi}{c}K(\hat{x}\times\hat{z}) = -\frac{4\pi}{c}K\hat{y}$$

$$\Rightarrow \quad b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi}{c}K$$

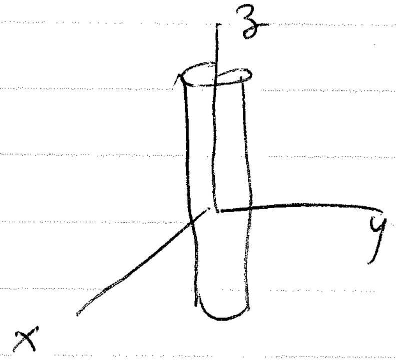
define $\left.\begin{array}{l} b_y^> = b_{y0} + \delta b_y \\ b_y^< = b_{y0} - \delta b_y \end{array}\right\} \quad \delta b_y = -\frac{2\pi}{c}K$

$$\Rightarrow \quad \vec{B}^> = \vec{B}_0 - \frac{2\pi}{c}K\hat{y} \qquad \vec{B}_0 = b_{x0}\hat{x} + b_{y0}\hat{y} + b_{z0}\hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c}K\hat{y}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c}K\hat{y} & z>0 \\ \frac{2\pi}{c}K\hat{y} & z<0 \end{cases}$$

example current carrying infinite cylinder radius R



(i) $\vec{K} = K\hat{z}$ wire with surface current
(ii) $\vec{K} = K\hat{\varphi}$ solenoid

(i) $\vec{K} = K\hat{z}$ $2\pi R K = I$ total current

"guess" + show it is correct

$r>R$ $\boxed{\phi_M = -\frac{4\pi}{c} R K \varphi}$ magnetic scalar potential $\nabla^2 \phi_M = 0$

$r<R$ $\phi_M = 0$

$r>R$ $\vec{B} = -\vec{\nabla}\phi_M = -\frac{1}{r}\frac{\partial \phi_M}{\partial \varphi}\hat{\varphi} = \frac{4\pi R K}{c r}\hat{\varphi} = \boxed{\frac{2I}{cr}\hat{\varphi}}$ familiar result from Ampere

$r<R$ $\vec{B} = 0$

$\vec{B}_{above} - \vec{B}_{below} = \frac{2I}{cR}\hat{\varphi} = \frac{4\pi K}{c}\frac{R}{R}\hat{\varphi} = \frac{4\pi}{c}\vec{K}\times\hat{n}$
where $\hat{n} = \hat{r}$
as $\hat{z}\times\hat{r} = \hat{\varphi}$

Note: $\phi_M = -\frac{4\pi}{c} R K \varphi$ is not single valued!
would not have found this using expansion
of separation of coords in polar coords
ϕ_M does not need to be single valued since it has
no physical significance. Only $\vec{B} = -\vec{\nabla}\phi_M$ is physical

(ii) $\vec{K} = K\hat{\varphi}$

$r>R$ $\phi_M = -B_1 z$
$r<R$ $\phi_M = -B_2 z$

$\Big\}\ \nabla^2\phi_M = 0$

$r>R$ $\vec{B} = -\vec{\nabla}\phi_M = B_1\hat{z}$
$r<R$ $\vec{B} = -\vec{\nabla}\phi_M = B_2\hat{z}$

$$\vec{B}_{above} - \vec{B}_{below} = (B_1 - B_2)\,\hat{z} = \frac{4\pi}{c}\vec{K} \times \hat{n}$$
$$= \frac{4\pi}{c} K\,(\hat{\varphi} \times \hat{r})$$
$$= -\frac{4\pi}{c} K\,\hat{z}$$

If current in solenoid is only source of $\vec{B}$ Then expect $B_1 = 0$

$$\rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c} K\,\hat{z}} \checkmark$$ familiar result