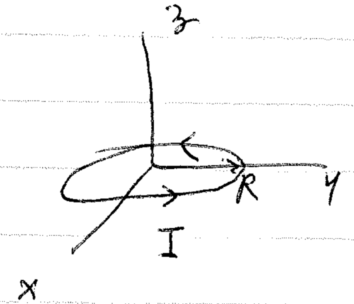
<u>example</u> <u>circular current loop</u> in xy plane
radius R



for $r > R$, $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla}\phi_M$
where $\nabla^2 \phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

(A_ℓ terms vanish as want $B \to 0$ as $r \to \infty$)

$$\vec{B} = -\vec{\nabla}\phi_M = -\frac{\partial \phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi_M}{\partial \theta}\hat{\theta}$$

$$= \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} - \frac{B_\ell}{r^{\ell+2}} \frac{\partial P_\ell(\cos\theta)}{\partial\theta}\,\hat{\theta} \right]$$

write $\dfrac{\partial P_\ell}{\partial\theta} = \dfrac{\partial P_\ell}{\partial x}\dfrac{\partial x}{\partial\theta} = -\dfrac{\partial P_\ell}{\partial x}\sin\theta \equiv -P_\ell' \sin\theta$ $\quad x = \cos\theta$

$$\vec{B} = \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} + \frac{B_\ell}{r^{\ell+2}} \sin\theta\, P_\ell'(\cos\theta)\,\hat{\theta} \right]$$

To determine the B_ℓ we compare with exact solution along $\hat{z}$ axis

$$\vec{B}(z\hat{z}) = \sum_\ell \frac{(\ell+1)B_\ell}{r^{\ell+2}}\hat{r} = \sum_\ell \frac{(\ell+1)B_\ell}{z^{\ell+2}}\hat{z}$$

since $P_\ell(1) = 1$, $\sin(0) = 0$ and $P_\ell'(1)$ finite, $\hat{r} = \hat{z}$ when $\theta = 0$

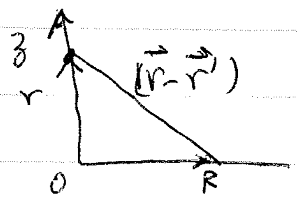
exact solution on $\hat{z}$ axis:

$$\vec{A} = \int \frac{d^3r'}{c} \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \int \frac{d^3r'}{c} \vec{\nabla}\times \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = -\int \frac{d^3r'}{c} \vec{J}(\vec{r}') \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\vec{B} = \int \frac{d^3r'}{c} \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Biot-Savart Law for magnetostatics

For our loop



polar radial vector

$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I\hat{\phi} \times \frac{[-R\hat{r} + z\hat{z}]}{(z^2+R^2)^{3/2}}$$

$\hat{r}\times\hat{\phi} = \hat{z}$

$$= \int_0^{2\pi} \frac{d\phi}{c} \frac{R(IR)\hat{z}}{(z^2+R^2)^{3/2}}$$ $\hat{\phi}\times\hat{z}$ term integrates to zero

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c(z^2+R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c z^3} \frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I \hat{z}}{z^3}\left\{1 - \frac{3}{2}\left(\frac{R}{z}\right)^2 + \cdots\right\}$$

$$= \frac{2\pi R^2 I}{c}\hat{z}\left\{\frac{1}{z^3} - \frac{3}{2}\frac{R^2}{z^5} + \cdots\right\}$$

$$= \left\{\frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \cdots\right\}\hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c}\pi R^2 I R^2$$

So to order $\ell = 3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2P_1(\cos\theta)\hat{r} + \sin\theta P_1'(\cos\theta)\hat{\theta}}{r^3} \right.$$

$$\left. - \left[\frac{3R^2 P_3(\cos\theta)\hat{r} + \frac{3}{4}R^2 \sin\theta P_3'(\cos\theta)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

$$P_1(x) = x \quad \Rightarrow \quad P_1'(x) = 1$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \tfrac{1}{2}(15x^2 - 3)$$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3} \right.$$

$$\left. - \left[\frac{\frac{3}{2}R^2(5\cos^3\theta - 3\cos\theta)\hat{r} + \frac{3}{8}R^2\sin\theta(15\cos^2\theta - 3)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

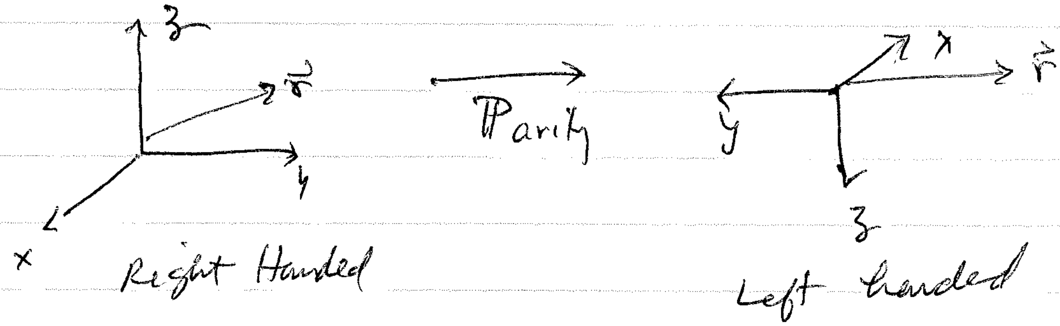
$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic ~~quadrupole~~ octopole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5-40)

Symmetry under parity transformation

vector vs. pseudo vector



$$\vec{r} = (x, y, z) \longrightarrow (-x, -y, -z)$$

$$\mathbb{P}(\vec{r}) = -\vec{r}$$ position $\vec{r}$ is <u>odd</u> under parity

Any vector-like quantity that is odd under $\mathbb{P}$ is a <u>vector</u>.

<u>examples of vectors</u>

position $\vec{r}$

velocity $\vec{v} = \frac{d\vec{r}}{dt}$ since $\vec{r}$ is vector and t is scalar $\mathbb{P}(t) = t$

acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

Force $\vec{F} = m\vec{a}$ since $\vec{a}$ is vector and m is scalar

momentum $\vec{p} = m\vec{v}$ since $\vec{v}$ is vector and m is scalar

electric field $\vec{F} = q\vec{E}$ since $\vec{F}$ is vector and q is scalar $\mathbb{P}(q) = q$

current $\vec{J} = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i(t))$

any vector-like quantity that is <u>even</u> under $\mathbb{P}$ is a <u>psuedovector</u>

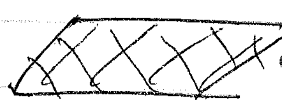
angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \to -\vec{r}$ and $\vec{p} \to -\vec{p}$,
$\vec{L} \to \vec{L}$ under $\mathbb{P}$
$\vec{L}$ is even under $\mathbb{P}$

magnetic field $\vec{F} = q\vec{v} \times \vec{B}$ since $\vec{F}$ and $\vec{v}$ are vectors and q is scalar, $\vec{B}$ must be ~~psudov~~ pseudovector.

cross product of any two vectors is a psuedovector
" " " vector and pseudovector is a vector

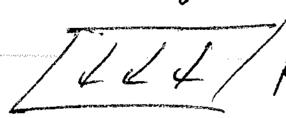
When solving for $\vec{E}$, it can only ~~be~~ made up of <u>vectors</u> that exist in the problem
When solving for $\vec{B}$, it can only be made up of <u>psuedovectors</u> that exist in the problem

<u>ex</u> charged plane


σ

only directions in problem is normal $\hat{n}$
$\hat{n}$ is a <u>vector</u>
$\vec{E} \propto \hat{n}$

surface current


K

only directions are the <u>vectors</u> $\hat{n}$ and $\vec{K}$. But $\vec{B}$ can only be made of <u>psuedovectors</u>

$$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{n})$$

Dielectrics + Magnetic Materials - Macroscopic Maxwell Equ

~~Dielectrics~~

Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents

$$\vec{\nabla}\cdot\vec{b} = 0 \qquad \vec{\nabla}\times\vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho_0 \qquad \vec{\nabla}\times\vec{b} = \frac{4\pi}{c}\vec{j}_0 + \frac{1}{c}\frac{\partial\vec{e}}{\partial t}$$

where $\vec{e}$ and $\vec{b}$ are microscopic fields from total charge density ρ_0 and current density $\vec{j}_0$.

However, in most problems involving macroscopic objects, if we took ρ_0 and $\vec{j}_0$ to describe charge + current of each individual atom in a material, then they, and the resulting $\vec{e}$ and $\vec{b}$ would be enormously complicated functions varying rapidly over distances $\sim 10^{-8}$ cm and times $\sim 10^{-16}$ sec.

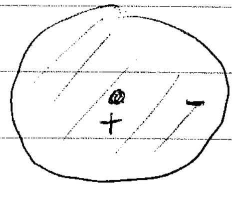
In classical E+M we are generally concerned with phenomena that vary extremely slowly compared to these length + time scales, Rather than worry about the microscopic details of ρ and $\vec{j}$ and resulting $\vec{e}$ and $\vec{b}$ we want to describe phenomena in terms of ~~averaged~~ ~~smoothly varying quan~~ averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the <u>macroscopic</u> <u>Maxwell equations</u>.

<u>Dielectric Materials</u>

can be solid, liquid or gas

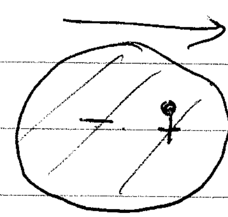
A dielectric material is an insulator. valence Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged ρ in the dielectric <u>vanishes</u>! One might therefore think that electrodynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However this is not true due to the phenomena of "polarization".

$E = 0$



electron cloud centered on ionic ~~nucle~~ core
dipole moment vanishes

$E > 0$



electron cloud and ionic core displaced $\vec{d} \propto \vec{E}$
atom is "polarized"
has dipole moment $\vec{p} = q\vec{d} \propto q\vec{E}$

$\vec{p} = \alpha \vec{E}$

α atomic polarizability