

Macroscopic Maxwell Equations

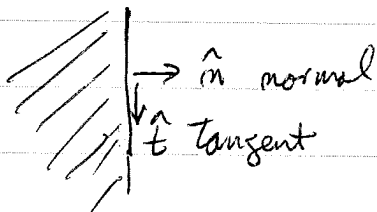
$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \times \vec{H} &= \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} & \vec{\nabla} \cdot \vec{D} &= 4\pi \rho\end{aligned}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$$\begin{aligned}\vec{D} &= \vec{E} + 4\pi \vec{P}, & \vec{P} &\text{ is polarization density} \\ \vec{H} &= \vec{B} - 4\pi \vec{M}, & \vec{M} &\text{ is magnetization density}\end{aligned}$$

Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics



$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \hat{t} \cdot \vec{E}_{\text{above}} = \hat{t} \cdot \vec{E}_{\text{below}}$$

tangential component $\vec{E}$ is continuous

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho \Rightarrow \hat{n} \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = 4\pi \sigma$$

normal component of $\vec{D}$ jumps by $4\pi \sigma$

magnetostatics: at surface or interface of magnetic materials

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \hat{n} \cdot \vec{B}_{\text{above}} - \hat{n} \cdot \vec{B}_{\text{below}}$$

normal component of $\vec{B}$ is continuous

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} \Rightarrow \hat{t} \cdot (\vec{H}_{\text{above}} - \vec{H}_{\text{below}}) = \frac{4\pi}{c} (\vec{K} \times \hat{n}) \cdot \hat{t}$$

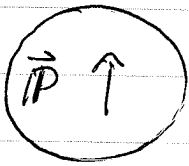
tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c} \vec{K} \times \hat{n}$

if $\sigma = 0$, i.e. no free surface charge, then $\hat{n} \cdot \vec{D}$ continuous

if $\vec{K} = 0$, i.e. no free surface current, then $\hat{t} \cdot \vec{H}$ continuous

Examples

① Uniformly polarized sphere of radius R $\vec{P} = P \hat{z}$



bound charge $\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ as $\vec{P}$ constant

$$\sigma_b = \hat{n} \cdot \vec{P} = \hat{r} \cdot \vec{P} = P \cos \theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0 \cos \theta$ gives an electric field like a pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3} \pi R^3 P \right) \left[\frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right] & r > R \\ -\frac{4\pi P}{3} \hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3} \pi R^3 \vec{P}$

check behavior at boundary

Tangential component $\vec{E}$

$$\vec{E}_{\text{above}}^{\perp} = \left(\frac{4}{3} \pi R^3 P \right) \frac{\sin \theta \hat{\theta}}{R^3} = \frac{4\pi P}{3} \sin \theta \hat{\theta}$$

$$\vec{E}_{\text{below}}^{\perp} = -\frac{4\pi P}{3} (\hat{z} \cdot \hat{\theta}) \hat{\theta} = \frac{4\pi P}{3} \sin \theta \hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

normal component of $\vec{D}$

outside: $\vec{P} = 0 \Rightarrow \vec{D} = \vec{E}$

$$\Rightarrow \hat{n} \cdot \vec{D} = \hat{r} \cdot \vec{E} = \left(\frac{4}{3} \pi R^3 P \right) \frac{2 \cos \theta \hat{r}}{R^3} = \frac{8\pi P}{3} \cos \theta$$

Inside: $\vec{E} = -\frac{4\pi\vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3}P\hat{z}$$

$$\hat{n} \cdot \vec{D} = \hat{r} \cdot \left(\frac{8\pi}{3}P\hat{z} \right) = \frac{8\pi}{3}P\cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi P\cos\theta$ inside

to check this: $\hat{n} \cdot \vec{E} = \hat{r} \cdot \left(-\frac{4}{3}\pi P\hat{z} \right) = -\frac{4}{3}\pi P\cos\theta$

$$\hat{n} \cdot (\vec{E}^{\text{above}} - \vec{E}^{\text{below}}) = \frac{8}{3}\pi P\cos\theta + \frac{4}{3}\pi P\cos\theta$$

$$= \frac{12}{3}\pi P\cos\theta = 4\pi P\cos\theta = \sigma_b(\theta)$$

② Uniformly magnetized sphere of radius R $\vec{M} = M \hat{z}$



bound current $\vec{J}_b = c \vec{\nabla} \times \vec{M} = 0$ as $\vec{M}$ constant
 $\vec{K}_b = c \vec{M} \times \hat{n} = cM (\hat{z} \times \hat{r})$
 $= cM \sin \theta \hat{\phi}$

We saw earlier that a sphere with surface current $\vec{K}_b = K_0 \sin \theta \hat{\phi}$ gives a magnetic field that is pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3} \pi R^3 M \right) \left[\frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right] & r > R \\ \frac{8}{3} \pi M \hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3} \pi R^3 \vec{M}$

check behavior at boundary

normal component of $\vec{B}$

$$\hat{m} \cdot \vec{B}^{\text{above}} = \hat{r} \cdot \vec{B}^{\text{above}} = \frac{8}{3}\pi M \cos\theta$$

$$\hat{m} \cdot \vec{B}^{\text{below}} = \hat{r} \cdot \vec{B}^{\text{below}} = \frac{8}{3}\pi M (\hat{r} \cdot \hat{z}) = \frac{8}{3}\pi M \cos\theta$$

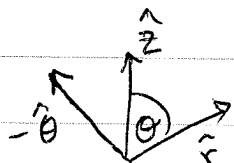
$\Rightarrow$ normal component of $\vec{B}$ is continuous

tangential component of $\vec{H}$

outside: $\vec{M} = 0 \Rightarrow \vec{H} = \vec{B}$

$$\vec{H}_{\text{above}}^t = \left(\frac{4}{3}\pi M\right) \sin\theta \hat{\theta}$$

inside: $\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$
 $= -\frac{4\pi}{3}M\hat{z}$



$$\text{so } \vec{H}_{\text{below}}^t = -\frac{4\pi}{3}M(\hat{z} \cdot \hat{\theta})\hat{\theta} = \frac{4\pi}{3}M \sin\theta \hat{\theta}$$

$\Rightarrow$ tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b \times \hat{m} = 4\pi M \sin\theta \hat{\theta}$

inside:
to check: $\vec{B}_{\text{below}}^t = \frac{8}{3}\pi M(\hat{z} \cdot \hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M \sin\theta \hat{\theta}$

$$\begin{aligned} \vec{H}_{\text{above}}^t = \vec{B}_{\text{above}}^t &\Rightarrow \vec{B}_{\text{above}}^t - \vec{B}_{\text{below}}^t = \frac{4}{3}\pi M \sin\theta \hat{\theta} + \frac{8}{3}\pi M \sin\theta \hat{\theta} \\ &= 4\pi M \sin\theta \hat{\theta} = \frac{4\pi}{c}\vec{K}_b \end{aligned}$$

Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge & current densities
and

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

$\vec{P}$ is polarization density

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$\vec{M}$ is magnetization density

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

Ferrromagnet: $\vec{M}$ can be non zero even if $\vec{B}=0$

Ferroelectric: $\vec{P}$ can be non zero even if $\vec{E}=0$

But more common are linear materials in which, for small $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$$\vec{P} = \chi_e \vec{E}$$

χ_e is "electric susceptibility"
 $\chi_e > 0$ for statics

$$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$$

$$\vec{D} = \epsilon \vec{E} \quad \text{with} \quad \epsilon = 1 + 4\pi \chi_e$$

ϵ is the dielectric constant

linear magnetic materials

$$\vec{M} = \chi_m \vec{H}$$

χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$$

$$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$$

$$\vec{B} = \mu \vec{H} \quad \text{with} \quad \mu = 1 + 4\pi \chi_m$$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.

Clausius - Mossotti equation

Electric susceptibility & atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

↑ ↑ ↑
atomic dipole moment atomic polarizability "local field" - field the atom sees

α is what one calculates from a microscopic theory

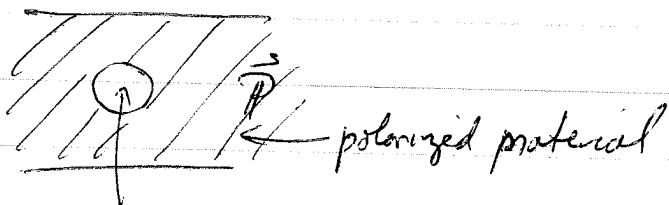
If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

$$\vec{P} = n \vec{p} = n \alpha \vec{E}_{loc} = n \alpha \vec{E} = \chi_e \vec{E}$$
$$\Rightarrow \chi_e = n \alpha \quad \text{where } n = \text{density of atoms}$$

But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$
The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

↑ ↑ ↑
average field average field excluding atom average field of the atom



cut out sphere whose volume is $1/n$
the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$\vec{E}_{atom} = -\frac{4\pi\vec{P}}{3} = -\frac{4\pi}{3}m\vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi\vec{P}}{3} = \vec{E} + \frac{4\pi}{3}m\vec{p}$$

$$\vec{p} = \alpha \vec{E}_{loc} = \alpha \left(\vec{E} + \frac{4\pi}{3}m\vec{p} \right) = \alpha \vec{E} + \frac{4\pi m \alpha}{3} \vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi}{3}m\alpha} \vec{E}$$

$$\vec{P} = m\vec{p} = \frac{\alpha m}{1 - \frac{4\pi}{3}m\alpha} \vec{E} = \chi_e \vec{E}$$

$$\chi_e = \frac{m\alpha}{1 - \frac{4\pi}{3}m\alpha}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3}n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3}n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n(1 + \frac{4\pi}{3}\chi_e)}$$

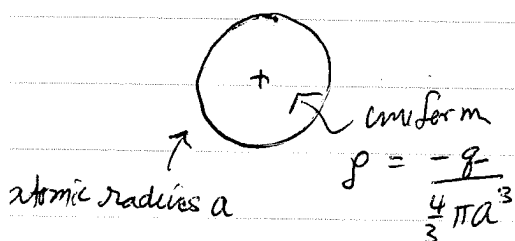
$$\epsilon = 1 + 4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon - 1}{4\pi n} \frac{1}{(1 + \frac{\epsilon - 1}{3})}$$

relates atomic
polarizability to
measured dielectric constant

$$\alpha = \frac{3}{4\pi n} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)$$

Clausius Mossotti
or Lorentz-Lorenz equation

single model for α



field inside is $E(r) = \frac{4\pi}{3} \rho r \hat{r}$

plus minus

In external field E_0 , net forces
balance $\Rightarrow qE_0 = q \frac{4\pi}{3} \rho d$

$$\chi_e = \frac{na^3}{1 - \frac{4\pi}{3}na^3}$$

$$p = qd = \frac{3}{4\pi \rho} q E_0 = \frac{3}{4\pi \rho} (4\pi \rho a^3) q E_0$$

$$= a^3 E_0 \Rightarrow \alpha = a^3$$

if $f = n \frac{4\pi}{3} a^3$ fraction of vol that is occupied by atoms

$$\chi_e = \frac{1}{4\pi} \frac{3f}{1-f}$$

Linear Dielectrics

bound charge is proportional to free charge

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot (\chi_e \vec{E}) = -\vec{\nabla} \cdot \left(\frac{\chi_e}{\epsilon} \vec{D} \right)$$

if χ_e (and hence ϵ) is spatially constant, then

$$\rho_b = -\frac{\chi_e}{\epsilon} \vec{\nabla} \cdot \vec{D} = -\frac{\chi_e}{\epsilon} 4\pi \rho$$

$$\boxed{\rho_b = -\frac{4\pi\chi_e}{1+4\pi\chi_e} \rho}$$

when free charge $\rho = 0$,
then $\rho_b = 0$

$$\rho_{\text{total}} = \rho + \rho_b = \rho \left[1 - \frac{4\pi\chi_e}{1+4\pi\chi_e} \right] = \frac{\rho}{1+4\pi\chi_e} = \boxed{\frac{\rho}{\epsilon} = \rho_{\text{total}}}$$

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.