

Magnetostatics

Bar magnets - $\vec{j} = 0$, $\vec{M}$ fixed and given
(not a linear material)

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} = 0$$

$$\vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla} \Phi_M \quad \text{magnetic scalar potential}$$

$$\vec{B} = \vec{H} + 4\pi \vec{M}$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{H} + 4\pi \vec{M}) = 0$$

$$\vec{\nabla} \cdot \vec{H} = -\nabla^2 \Phi_M = -4\pi \vec{\nabla} \cdot \vec{M}$$

$$\nabla^2 \Phi_M = 4\pi \vec{\nabla} \cdot \vec{M}$$

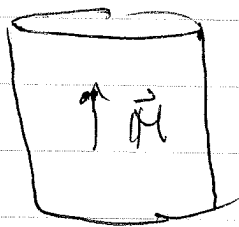
so $\rho_M \equiv -\vec{\nabla} \cdot \vec{M}$ looks like a magnetic "charge"

ρ_M is source for $\vec{H}$

also at surfaces of material $\sigma_M = \hat{n} \cdot \vec{M}$ looks like surface charge

$$\vec{H}(\vec{r}) = \int_V d^3r' \rho_M(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} + \oint_S da' \sigma_M(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Field lines for $\vec{H}$ can start and end at sources and sinks given by ρ_M and σ_M



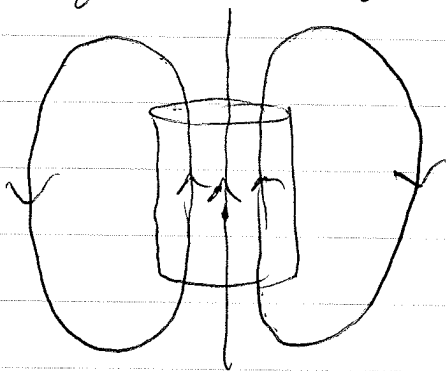
$$\vec{M} = M \hat{z}$$

bound currents $\vec{j}_b = c \vec{\nabla} \times \vec{M} = 0$

$$\vec{K}_b = c \vec{M} \times \hat{n}$$

$$\vec{K}_b = \begin{cases} cM \odot & \text{on side} \\ 0 & \text{on top + bottom} \end{cases}$$

$\vec{K}_b$ is like solenoid current
field lines of $\vec{B}$ look like

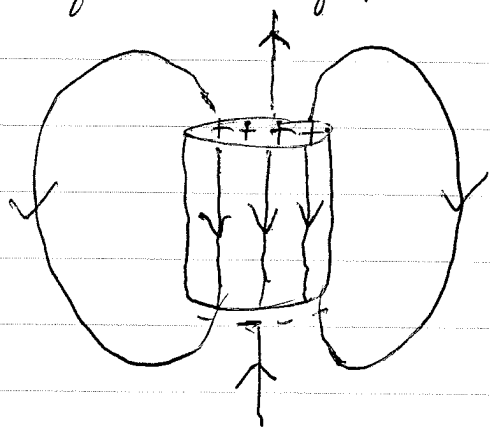


But $\vec{H}$ is determined as follows:

$$\oint_M = -\vec{\nabla} \cdot \vec{M} = 0$$

$$\sigma_M = \vec{n} \cdot \vec{M} = \begin{cases} M & \text{on top} \\ -M & \text{on bottom} \end{cases}$$

field lines of $\vec{H}$ look like parallel plate capacitor



field lines of $\vec{H}$ = field lines of $\vec{B}$
outside magnet, but they
are very different inside
the magnet!

Conservation of Energy

- leave macroscopic Maxwell eqs for present. $\vec{E}, \vec{B}, \rho, \vec{J}$ are now the exact microscopic quantities

Consider a collection of charged particles, described by charge density ρ and current density $\vec{J}$. The particles are contained in a volume V .

Define E_{mech} as total "mechanical" energy of the particles. E_{mech} = sum of particles kinetic energy plus potential energy of any non electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, i.e. via the $\vec{E}$ and $\vec{B}$ fields that they create. Define W as the work done on the particles by all electromagnetic forces. Then, by the work energy theorem of mechanics:

$$\frac{dE_{\text{mech}}}{dt} = \frac{dW}{dt}$$

For a single charge q_i , $\frac{dW}{dt} = \vec{F}_i \cdot \vec{v}_i$
(at $\vec{r}_i$ with velocity $\vec{v}_i$)

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i + q_i \left(\frac{\vec{v}_i \times \vec{B}}{c} \right) \cdot \vec{v}_i$$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i \quad \quad \quad 0$$

For the collection of charges, with

$$\vec{j}(\vec{r}, t) = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r \vec{j} \cdot \vec{E}$$

So

$$\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{j} \cdot \vec{E}$$

By maxwell equation $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$
we can write

$$\vec{j} = \frac{c}{4\pi} \left[\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right]$$

$$\int_V d^3r \vec{j} \cdot \vec{E} = \int_V d^3r \frac{c}{4\pi} \left[(\vec{\nabla} \times \vec{B}) \cdot \vec{E} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \right]$$

$$\text{use } \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} = \frac{1}{2} \frac{\partial E^2}{\partial t}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\text{then use } \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\text{So } \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$= -\frac{1}{2c} \frac{\partial B^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

Combine results to get

$$\int_V d^3r \vec{f} \cdot \vec{E} = -\frac{1}{4\pi} \int_V d^3r \left[\frac{1}{2} \frac{\partial B^2}{\partial t} + \frac{1}{2} \frac{\partial E^2}{\partial t} + c \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right]$$

define $u = \frac{1}{8\pi} (E^2 + B^2)$ electromagnetic energy density

then $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$ Poynting vector - energy current

$$\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{f} \cdot \vec{E} = - \int_V d^3r \left[\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right]$$

If we define E_{EM} , the electromagnetic energy of the volume V , as

$$E_{\text{EM}} = \int_V d^3r u$$

then

$$\frac{d}{dt} (E_{\text{mech}} + E_{\text{EM}}) = - \oint_S da \hat{n} \cdot \vec{S}$$

or we write $\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{f} \cdot \vec{E}$ as the rate of change of mechanical energy

or we can write in differential form

$$\vec{f} \cdot \vec{E} + \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

↑
rate of change of mechanical energy per unit volume

local energy conservation law
if interpret $\vec{S}$ as energy current and u as EM energy density

$$\frac{d}{dt} (\mathcal{E}_{\text{mech}} + \mathcal{E}_{\text{EM}}) = - \oint_S da \hat{n} \cdot \vec{S}$$

total energy in V can decrease only if electromagnetic energy is being transported through the surface S by the EM energy current $\vec{S}$.

assumes the charged particles do not leave the volume V .

under certain conditions, we can derive a similar conservation law for the macroscopic maxwell eqs.

Consider that $\vec{j}$ is current of the free ^{charged} particles.

Then repeating the above steps:

$$\int_V d^3r \vec{j} \cdot \vec{E} = \frac{c}{4\pi} \int d^3r \vec{E} \cdot \left[\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \right]$$

$$\begin{aligned} \vec{\nabla} \cdot (\vec{E} \times \vec{H}) &= \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) \\ &= -\frac{1}{c} \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{\nabla} \times \vec{H} \end{aligned}$$

so

$$\int_V d^3r \vec{j} \cdot \vec{E} = -\frac{1}{4\pi} \int_V d^3r \left[c \vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

If the medium is linear, and we have quasistatic conditions, so that

$$\vec{D}(t) \approx \epsilon \vec{E}(t)$$

$$\vec{H}(t) \approx \frac{1}{\mu} \vec{B}(t)$$

then we can write

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$$

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{\mu} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2\mu} \frac{\partial B^2}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H})$$

**** Note:** in general, as we will soon see, above conditions are not satisfied, ϵ will depend on frequency ω and $\vec{D}(t)$ and $\vec{E}(t)$ are non locally related in time.
 $\vec{D}(t) = \int_{-\infty}^{\infty} dt' \epsilon(t-t') \vec{E}(t')$. Only at low frequencies,

in quasistatic case, can we write $\vec{D}(t) \approx \epsilon(\omega=0) \vec{E}(t)$.

Assuming the above conditions are met, then

$$\int_V d^3r \vec{j} \cdot \vec{E} + \int_V d^3r \frac{\partial u}{\partial t} = - \oint_S da \hat{n} \cdot \vec{S}$$

$$\text{where } \begin{cases} u = \frac{1}{8\pi} [\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}] \\ \vec{S} = \frac{c}{4\pi} [\vec{E} \times \vec{H}] \end{cases}$$

$\Rightarrow$ electromagnetic energy in dielectric + magnetic materials under quasistatic conditions is

$$\int_V d^3r \left[\underbrace{\frac{1}{8\pi} \vec{E} \cdot \vec{D}}_{\text{electrostatic energy}} + \underbrace{\frac{1}{8\pi} \vec{B} \cdot \vec{H}}_{\text{magnetostatic energy}} \right]$$

Statics

Electrostatic Energy

Returning to microscopic fields and charges

$$\mathcal{E} = \frac{1}{8\pi} \int_V d^3r E^2 \quad \text{use } \vec{E} = -\vec{\nabla}\phi$$

$$= -\frac{1}{8\pi} \int_V d^3r (\vec{\nabla}\phi) \cdot \vec{E} \quad \text{use } \vec{\nabla} \cdot (\phi \vec{E}) = \phi \vec{\nabla} \cdot \vec{E} + (\vec{\nabla}\phi) \cdot \vec{E}$$

$$= -\frac{1}{8\pi} \int_V d^3r [\vec{\nabla} \cdot (\phi \vec{E}) - \phi \vec{\nabla} \cdot \vec{E}] \quad \text{use } \vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$= \frac{1}{2} \int_V d^3r \rho \phi - \frac{1}{8\pi} \oint_S da \hat{n} \cdot \phi \vec{E} \quad \text{by Gauss Theorem}$$

If let V be all space, $S \rightarrow \infty$, then $\phi \sim \frac{1}{r}$, $E \sim \frac{1}{r^2}$
surface integral $\sim \frac{R^2}{R^3} \rightarrow 0$ as $R \rightarrow \infty$.

$$\boxed{\mathcal{E} = \frac{1}{2} \int d^3r \rho \phi}$$

can also use $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$ to write

$$\boxed{\mathcal{E} = \frac{1}{2} \int d^3r d^3r' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}}$$

charge - charge
interaction

Magnetostatic Energy

microscopic fields and currents

$$\mathcal{E} = \frac{1}{8\pi} \int d^3r B^2$$

$$\text{use } \vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \frac{1}{8\pi} \int d^3r \vec{B} \cdot \vec{\nabla} \times \vec{A}$$

$$\text{use } \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = \vec{A} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{A})$$

$$= \frac{1}{8\pi} \int d^3r \left[\vec{A} \cdot \vec{\nabla} \times \vec{B} - \vec{\nabla} \cdot (\vec{B} \times \vec{A}) \right] \quad \text{use } \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

$$= \frac{1}{2c} \int d^3r \vec{j} \cdot \vec{A} - \frac{1}{8\pi} \oint_S da \hat{n} \cdot (\vec{B} \times \vec{A})$$

as take V to fill all space, $S \rightarrow \infty$, surface term vanishes

$$\boxed{\mathcal{E} = \frac{1}{2c} \int d^3r \vec{j} \cdot \vec{A}}$$

$$\text{In Coulomb gauge } \vec{\nabla} \cdot \vec{A} = 0, \quad \vec{A}(\vec{r}) = \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

In any other gauge we have $\vec{A}' = \vec{A} + \vec{\nabla} \chi$
for some scalar χ . So we can always write

$$\vec{A}(\vec{r}) = \int d^3r' \frac{\vec{j}(\vec{r}')}{c |\vec{r} - \vec{r}'|} + \vec{\nabla} \chi$$

regardless of the choice of gauge, where χ is then determined so $\vec{A}$ satisfies the desired gauge condition

$$\mathcal{E} = \frac{1}{2c} \int d^3r d^3r' \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}')}{c |\vec{r} - \vec{r}'|} + \frac{1}{2c^2} \int d^3r \vec{j} \cdot \vec{\nabla} \chi$$

2nd term $\hookrightarrow \int d^3r \vec{j} \cdot \vec{\nabla} \chi = \int d^3r \left[\vec{\nabla} \cdot (\vec{j} \chi) - \chi \vec{\nabla} \cdot \vec{j} \right]$

$$= \oint_S da \hat{n} \cdot \vec{j} \chi - \int d^3r \chi \vec{\nabla} \cdot \vec{j}$$

vanishes as $S \rightarrow \infty$

vanishes in magnetostatics where $\vec{\nabla} \cdot \vec{j} = 0$

$\mathcal{E}_0$

$$\mathcal{E} = \frac{1}{2c^2} \int d^3r d^3r' \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

current-current interaction

Momentum Conservation

For charges q_i at positions $\vec{r}_i$ with velocities $\vec{v}_i$

$$\frac{d\vec{P}^{\text{mech}}}{dt} = \sum_i \vec{F}_i = \sum_i q_i (\vec{E}(\vec{r}_i) + \frac{1}{c} \vec{v}_i \times \vec{B}(\vec{r}_i))$$

$\uparrow$
 "mechanical"
 momentum of
 the charges

$\uparrow$
 force on
 charge i

$$= \int_V d^3r \left[\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \right]$$

$$\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} = \frac{1}{4\pi} \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) + (\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t}) \times \vec{B} \right]$$

$$\text{Now } \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{1}{c} \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) + \frac{1}{c} \left(\vec{E} \times \frac{\partial \vec{B}}{\partial t} \right) \quad \text{use } \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$= \frac{1}{c} \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) - \vec{E} \times (\vec{\nabla} \times \vec{E})$$

$$\text{So } -\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} = -\vec{E} \times (\vec{\nabla} \times \vec{E}) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

Therefore

$$\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} = \frac{1}{4\pi} \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) + \vec{B} (\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) \right]$$

$\downarrow = 0$

Define electromagnetic momentum density

$$\vec{\Pi} = \frac{1}{4\pi c} \vec{E} \times \vec{B} = \frac{1}{c^2} \vec{S} \quad (\vec{S} \text{ is Poynting vector})$$

then

$$\frac{d\vec{P}^{\text{mech}}}{dt} + \frac{d}{dt} \int_V d^3r \vec{\Pi} = \frac{1}{4\pi} \int_V d^3r \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) + \vec{B} (\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]$$

$\uparrow$
want to rewrite as a surface integral