## Harmonic

## Plane waves

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

complex exponential form

$\vec{k}$ is wave vector

$\omega$ is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \dfrac{2\pi}{|\vec{k}|}$ is wavelength

$\left.\begin{matrix}|\vec{E}_k| \\ |\vec{B}_k|\end{matrix}\right\}$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space with period $\lambda$

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

### properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E} = 0 \quad\Rightarrow\quad \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$
$$\Rightarrow \vec{E}_k\cdot\vec{k} = 0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad\Rightarrow\quad \vec{B}_k\cdot\vec{k} = 0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad Re\left[\vec{\nabla}\times \vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = Re\left[\frac{1}{c}\vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow Re\left[-\vec{B}_k \times \vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = Re\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \quad Re\left[i\vec{k}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = Re\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \quad \vec{k}\times\vec{B}_k = -\frac{\omega}{c}\vec{E}_k$$

$$\vec{k}\times\vec{k}\times\vec{B}_k = -k^2\vec{B}_k = -\frac{\omega}{c}\vec{k}\times\vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2}\vec{k}\times\vec{E}_k}$$

<u>Finally</u>

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k \nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \frac{\vec{E}_k}{c^2}\frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k(-k^2)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\omega^2}{c^2}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{c^2}$$

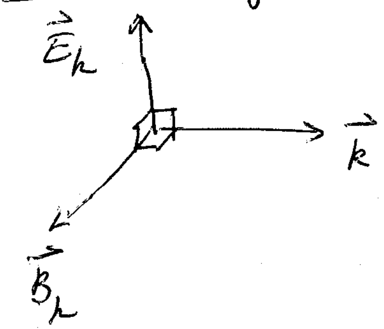
$$\boxed{\omega = \pm kc}$$ <u>dispersion relation</u>

insert in above

$$\vec{B}_k = \hat{k}\times\vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow \quad |\vec{B}_k| = |\vec{E}_k|$$

## Summary




$$\left.\begin{array}{l}\vec{E}_k \perp \vec{k} \\ \vec{B}_k \perp \vec{k}\end{array}\right\} \text{"transverse" polarization}$$

$$\vec{B}_k = \hat{k} \times \vec{E}_k$$

$$\omega^2 = c^2 k^2$$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge $q$ is

$$q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above harmonic plane waves

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_k\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{Fourier transform}$$

$$\vec{E}(\vec{r}, t) \text{ is real} \Rightarrow \vec{E}_k^{\,*} = \vec{E}_{-k}$$

For dispersion relation $\omega^2 = c^2 k^2$ we can write

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} - \vec{v}t)$$

where $\vec{v} = c\hat{k}$ is velocity of wave

If we ~~only~~ combine waves traveling in same direction $\hat{k}$, then

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_k\, e^{i\vec{k}\cdot(\vec{r} - \vec{v}t)} = \vec{E}(\vec{r} - \vec{v}t, 0)$$

The general plane wave solution of wave equation always has this property

$$\vec{E}(r, t) = \vec{E}(\vec{r} - \vec{v}t, 0)$$

If know $\vec{E}$ at $t=0$, then know $\vec{E}$ at all times $t$

Energy + momentum in EM wave

$$\vec{E} = Re\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$\vec{B} = Re\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \hat{k}\times\vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$
for real $\vec{E}_k$

energy density $\quad u = \frac{1}{8\pi}\left(E^2 + B^2\right)$

$$= \frac{1}{8\pi}\left[E_k^2 + E_k^2\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current $\quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$$= \frac{c}{4\pi}\left[\vec{E}_k \times (\hat{k}\times\vec{E}_k)\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi}\hat{k}\, E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c\, u\, \hat{k}$$

momentum density $\quad \vec{\Pi} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$

$$u = c\,|\vec{\Pi}|$$ — energy momentum relation of photons!

For visible light $\lambda \sim 5\times 10^{-7}\,m \sim 5000\,\text{Å}$

$$T = \frac{\lambda}{c} = 1.6\times 10^{-15}\,\text{sec}$$

most classical measurements on microscopic scales $\quad t \gg T, \quad \ell \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\ u = \frac{1}{8\pi} E_k^2 \qquad \text{as } \langle \cos^2\theta \rangle = 1/2$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal $\hat{n}$

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

# Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources
("free" charge and current vanishes)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

<u>linear materials</u>

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

if $\mu$ and $\epsilon$ were simply <u>constants</u> then the above would become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

Then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t}\right)$$
$$= -\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu\epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minor changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$ dispersion relation changed by constant factor

$$\vec{E}_k \perp \vec{k}$$
$$\vec{B}_k \perp \vec{k}$$

$$i \vec{k} \times \vec{E}_k = \frac{i\omega}{c} \vec{B}_k$$

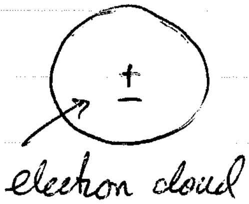
$$\frac{c|\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E}_k = \vec{B}_k \qquad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu\epsilon}} < c$

In general however, things are <u>much more complicated</u> because $\epsilon$ cannot be viewed as a constant when considering time varying behavior!

# Time dependent polarizeability of an atom




If displace center of electron cloud by a distance $\vec{r}$, there is a restoring force $\vec{F}_{rest} = \dfrac{-e^2\vec{r}}{4\pi R^3} \equiv -m\omega_0^2\vec{r}$

$m$ ↗ electron mass; $\omega_0$ ↗ resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{damp} = -m\gamma\frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = -\frac{e\vec{E}(t)}{m}$$

assume $\vec{E}$ is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2\vec{r}_0 - i\omega\gamma_0\vec{r}_0 + \omega_0^2\vec{r}_0 = -\frac{e\vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_0$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) E_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$ freq dependent polarizability

Since $\alpha$ is <u>complex</u> the polarization does not in general oscillate in phase with $\vec{E}$.

If $\alpha(\omega) = |\alpha| e^{i\delta}$  $\delta$ is phase of complex $\alpha$

$$\vec{p}(t) = \alpha(\omega)\vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

phase shifted by $\delta$

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} p_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $E_\omega = \int_{-\infty}^{\infty} dt' \, E(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \tilde{\alpha}(t-t')$$

Fourier transf of $\alpha(\omega)$

$\vec{p}$ at time $t$ is due to $\vec{E}$ at all times $t'$
non local in time

$\chi(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\chi}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

do by contour integration

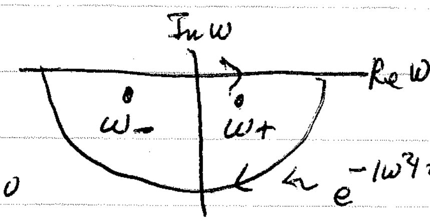
$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

poles at $\omega_\pm$ are in lower half complex plane.

for $t > 0$, close contour in lower half plane

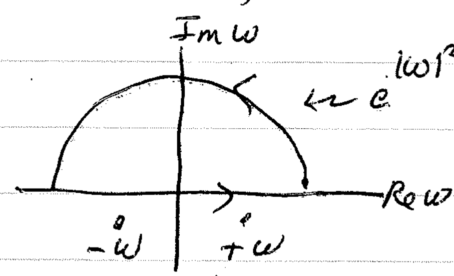
$t>0$




$\leftarrow e^{-|\omega|^2 t}$ here vanishes for $t>0$

contour encloses poles & get contribution

for $t < 0$, close contour in upper half plane




$\leftarrow e^{|\omega|^2 t}$ vanishes for $t<0$

contour encloses no poles $\Rightarrow$ integral vanishes

$\chi(t) = 0$ for $t < 0$

<u>causal response!</u> No polarization until electric field turns on