

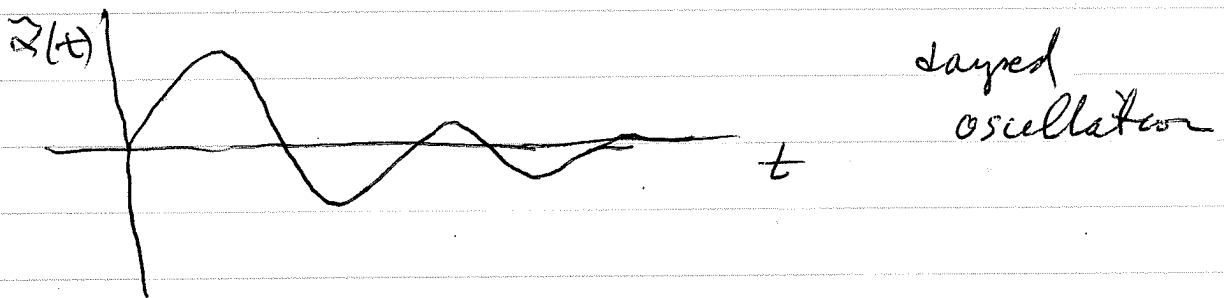
For $t > 0$

$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$= \underbrace{(-2\pi i)}_{\text{from residue theorem}} \frac{e^2}{m} \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

$$= \frac{ie^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



Polarization density $\vec{P}_\omega = 4\pi\chi(\omega)\vec{E}_\omega$ for harmonic oscillation

$\chi(\omega) \approx n\alpha(\omega)$ for dilute system

$\uparrow$ atom density

can use Clausius - Mossotti correction for denser materials

$$\Rightarrow \vec{D}_\omega = \epsilon(\omega)\vec{E}_\omega \quad \epsilon(\omega) = 1 + 4\pi\chi(\omega)$$

$\uparrow$ freq dependent

→ as with $\vec{f}$ and $\vec{E}$, relation between $\vec{D}$ and $\vec{E}$ is non-local in time

$$\vec{D}(t) \neq \epsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\epsilon}(t-t')$$

↑ Fourier transf of $\epsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{f} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

becomes $\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{f} + \frac{1}{c} \int_{-\infty}^{\infty} dt' \vec{E}(t') \frac{d}{dt} \tilde{\epsilon}(t-t')$

↑ integro-differential equation!

Maxwell's equations only look simple when expressed in terms of Fourier transforms

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{B}(\vec{r}, t) &= \vec{B}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{D}(\vec{r}, t) &= \vec{D}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{H}(\vec{r}, t) &= \vec{H}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned}$$

Maxwell's Equ for source free system $\rho = \vec{f} = 0$

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{c \partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{c \partial t}$$

assume μ is true constant - not freq dependent
dielectric response is $\vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

$$\begin{aligned} 1) \quad i \vec{k} \cdot \vec{D}_\omega &= i \epsilon(\omega) \vec{k} \cdot \vec{E}_\omega = 0 & \Rightarrow \boxed{\vec{k} \perp \vec{E}_\omega} \quad (\text{unless } \epsilon(\omega)=0) \\ 2) \quad i \vec{k} \cdot \vec{B}_\omega &= 0 & \Rightarrow \boxed{\vec{k} \perp \vec{B}_\omega} \\ 3) \quad i \vec{k} \times \vec{E}_\omega &= i \frac{\omega}{c} \vec{B}_\omega \\ 4) \quad i \vec{k} \times \vec{H}_\omega &= -i \frac{\omega}{c} \vec{D}_\omega \Rightarrow \frac{i \vec{k}}{\mu} \times \vec{B}_\omega = -i \frac{\omega}{c} \epsilon(\omega) \vec{E}_\omega \end{aligned}$$

$$\begin{aligned} \vec{k} \times (3) &= i \vec{k} \times (\vec{k} \times \vec{E}_\omega) = i \frac{\omega}{c} \vec{k} \times \vec{B}_\omega \\ &\Rightarrow -i k^2 \vec{E}_\omega = -i \frac{\omega^2}{c^2} \epsilon(\omega) \mu \vec{E}_\omega \quad \text{using (4)} \end{aligned}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2} \epsilon(\omega) \mu} \quad \text{dispersion relation}$$

~~$$\vec{B}_\omega \times \vec{E}_\omega = \frac{1}{\omega} \vec{k} \times \vec{E}_\omega \times \vec{E}_\omega = \frac{1}{\omega} \vec{k} \times (\vec{k} \times \vec{E}_\omega) = \frac{1}{\omega} \vec{k} \times \frac{c}{\omega} \vec{B}_\omega = \frac{c}{\omega^2} \vec{k} \times \vec{B}_\omega$$~~

Note: $\frac{\omega}{|\vec{k}|} = \frac{c}{\sqrt{\epsilon(\omega) \mu}}$

varies with ω .

there is not a single phase velocity.

$\Rightarrow \vec{E}$ is not in general a solution of a wave equation - different frequencies travel with different speeds

Since $\epsilon(\omega)$ is complex $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$

$\Rightarrow$ wave vector also complex For $\vec{k} = k \hat{z}$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_\omega e^{i[(k_1 + ik_2)z - \omega t]} \\ &= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}\end{aligned}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or attenuation of the wave as it propagates into the material

phase velocity $v_p = \frac{\omega}{k_1}$

index of refraction $n = \frac{c}{v_p} = \frac{ck_1}{\omega}$

group velocity v_g $\frac{1}{v_g} = \frac{dk_1}{d\omega}$

Magnetic field: $\vec{B}_\omega = \frac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k \hat{z}$, $\vec{B}_\omega = \frac{c(k_1 + ik_2)}{\omega} \hat{z} \times \vec{E}_\omega$

if $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2} e^{i\delta}$ $\delta = \arctan\left(\frac{k_2}{k_1}\right)$
 $= |k| e^{i\delta}$

$$\vec{B}_\omega = \frac{c|k|}{\omega} \hat{z} \times \vec{E}_\omega e^{i\delta}$$

$\uparrow$ phase shift

$$\vec{B}(\vec{r}, t) = \frac{c|k|}{\omega} (\hat{z} \times \vec{E}_\omega) e^{-k_2 z} e^{i(k_1 z - \omega t + \delta)}$$

Physical fields - take real parts

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r}, t) = (\hat{z} \times \vec{E}_\omega) \frac{c|k|}{\omega} e^{-k_2 z} \cos(k_1 z - \omega t + \delta)$$

Conclusions

- 1) $\vec{E}$ and $\vec{B} \perp \vec{k}$ transverse polarized
 - 2) $\vec{E} \perp \vec{B}$
 - 3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{\epsilon(\omega) \mu}$
 - 4) $\vec{B}$ is shifted in phase with respect to $\vec{E}$ by
phase shift $\delta = \arctan(k_2/k_1)$
 - 5) waves decay as they propagate $e^{-k_2 z}$
- } consequence of complex $\epsilon(\omega)$

If $\epsilon_2 = 0$, i.e. $\epsilon(\omega)$ is real, and if $\epsilon > 0$,
then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences
of
frequency
dependence
of $\epsilon(\omega)$

- 6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time
 - 7) waves of different ω travel with different $v_p = \omega/k_1$
 - 8) dispersion - wave pulses do not travel with v_p
and they spread as they propagate
pulses travel with group velocity $v_g = \frac{d\omega}{dk}$
(see Quantum Mechanics discussion)
- $v_g < v_p$ "normal dispersion"
 $v_g > v_p$ "anomalous dispersion"

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega} \left[\frac{\omega}{c} n \right]$$

index of refraction
↙

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c} \frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c} \frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c} \omega \frac{dn}{d\omega}}$$

$\Rightarrow$ when $\left\{ \begin{array}{l} \frac{dn}{d\omega} > 0, \quad v_g < v_p \text{ normal dispersion} \\ \frac{dn}{d\omega} < 0, \quad v_g > v_p \text{ anomalous dispersion} \end{array} \right.$

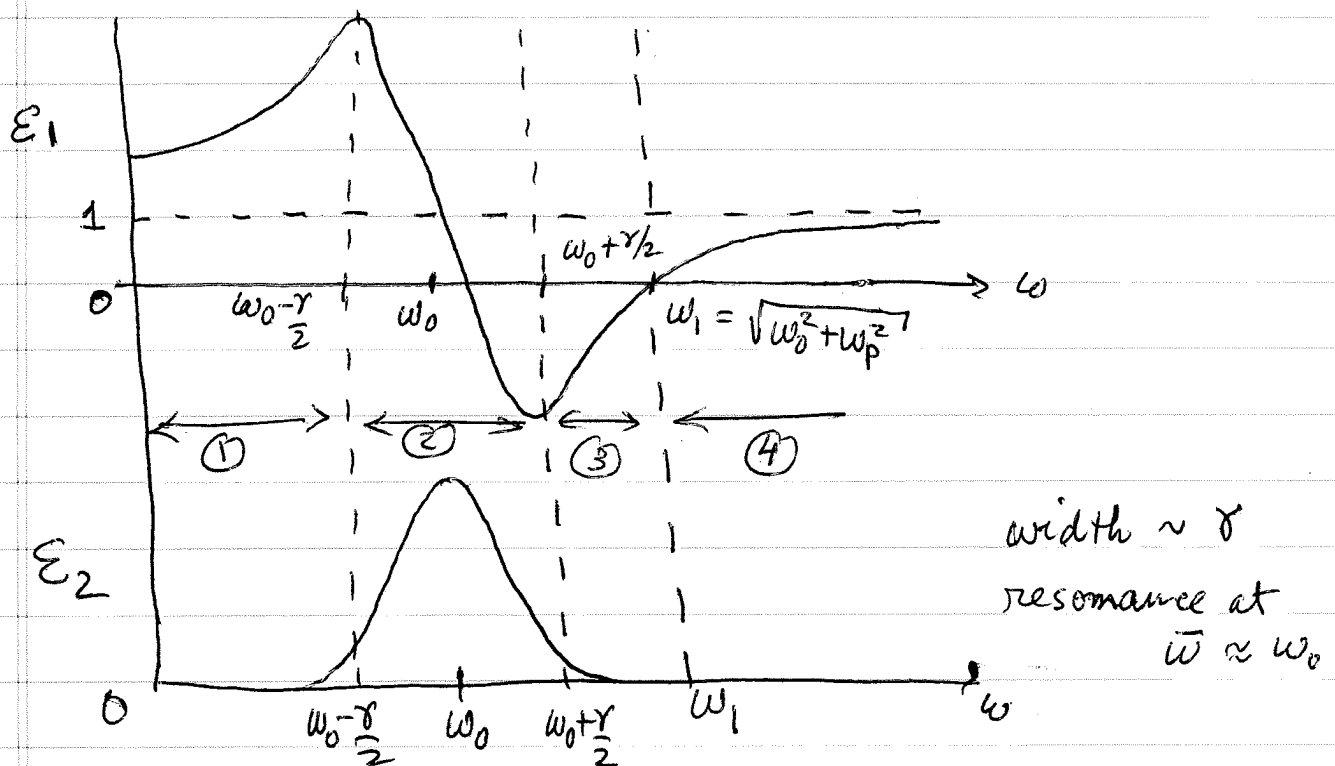
For our simple model: $\epsilon = 1 + 4\pi\chi \approx 1 + 4\pi\alpha$

$$\epsilon(\omega) = 1 + \frac{4\pi m e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\epsilon_1 = 1 + \frac{4\pi m e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

$$\epsilon_2 = \frac{4\pi m e^2}{m} \frac{\omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi m e^2}{m}}$ the "plasma frequency"



$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2} \mu (\epsilon_1 + i\epsilon_2)$$

equating real and imaginary pieces and
solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

Regions of different behavior

Regions ① and ④ - transparent propagation

$$\epsilon_1 > 0, \quad \epsilon_1 \gg \epsilon_2$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\epsilon_1 + \frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu \epsilon_1} + \text{small correction}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2} = k_1 \left(\frac{\epsilon_2}{2\epsilon_1} \right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation

$\Rightarrow$ medium is transparent

Note: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_1 \mu}}$

in region ①, $\epsilon_1 > 1 \Rightarrow v_p < c$

in region ④, $\epsilon_1 < 1 \Rightarrow v_p > c$!

but $v_g < c$ always !

Region ② $\omega \approx \omega_0$ resonant absorption

$$\epsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0} \right)^2 \left(\frac{\omega_0}{\gamma} \right) \gg 1 \quad \text{for a sharp resonance with } \gamma \ll \omega_0$$
$$\epsilon_1 \approx 1$$

So $\epsilon_2 \gg \epsilon_1$

$$k_1 \approx \pm \frac{\omega \sqrt{\mu}}{c} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$k_1 \approx k_2$ strong attenuation

wave excites atoms at resonance $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave $\Rightarrow$ wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

