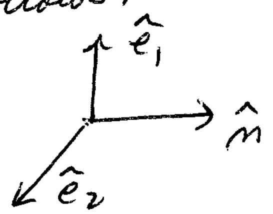
Polarization

Consider a transverse plane wave traveling in direction $\hat{n}$, ie $\vec{k} = k\hat{n}$. Define a right handed coordinate system as follows:



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution to Maxwell's equations for a transverse plane wave is then

$$\vec{E}(\vec{r},t) = Re\left\{ (E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$
$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{ k\,\hat{n}\times(E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$
$$= \frac{c}{\omega\mu} Re\left\{ k\,(E_1\hat{e}_2 - E_2\hat{e}_1)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

In general, k is complex

$$k = k_1 + i k_2 = |k|\, e^{i\delta}, \quad \begin{cases} |k| = \sqrt{k_1^2 + k_2^2} \\ \delta = \arctan(k_2/k_1) \end{cases}$$

So far we implicitly assumed that E_1 and E_2 are <u>real</u> constants. In this case

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t)$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t + \delta)$$

where

$$\vec{E}_\omega \equiv E_1\hat{e}_1 + E_2\hat{e}_2 \quad \text{and} \quad \vec{H}_\omega \equiv \frac{c|k|}{\omega\mu}(E_1\hat{e}_2 - E_2\hat{e}_1)$$

are <u>fixed</u> vectors for all time and space.

In this case the directions of $\vec{E}$ and $\vec{H}$ remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However there is nothing to prevent one from choosing a solution with E_1 and E_2 complex numbers,

$$E_1 = |E_1| e^{i\chi_1} \quad , \quad E_2 = |E_2| e^{i\chi_2}$$

In this case one has

$$\vec{E}(\vec{r},t) = Re\left\{ |E_1|\hat{e}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t+\chi_1)} + |E_2|\hat{e}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t+\chi_2)} \right\}$$

$$= e^{-k_2\hat{n}\cdot\vec{r}}\Big[|E_1|\hat{e}_1 \cos(k_1\hat{n}\cdot\vec{r}-\omega t+\chi_1) + |E_2|\hat{e}_2 \cos(k_1\hat{n}\cdot\vec{r}-\omega t+\chi_2)\Big]$$

and

$$\vec{H}(\vec{r},t) = \frac{c|k|}{\omega\mu} Re\left\{ |E_1|\hat{e}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta+\chi_1)} - |E_2|\hat{e}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta+\chi_2)} \right\}$$

$$= \frac{c|k|}{\omega\mu} e^{-k_2\hat{n}\cdot\vec{r}}\Big[|E_1|\hat{e}_2 \cos(k_1\hat{n}\cdot\vec{r}-\omega t+\delta+\chi_1) - |E_2|\hat{e}_1 \cos(k_1\hat{n}\cdot\vec{r}-\omega t+\delta+\chi_2)\Big]$$

Unless $\chi_1 = \chi_2$ we see that the components of $\vec{E}$ and $\vec{H}$ in directions $\hat{e}_1$ and $\hat{e}_2$ will oscillate out of phase with each other. Thus the directions of $\vec{E}$ and $\vec{H}$ will oscillate in time and space, as well as the amplitudes of $\vec{E}$ and $\vec{H}$. The direction of $\vec{E}$ and $\vec{H}$ is no longer fixed.

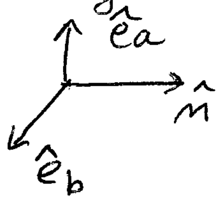
We will see that this situation in general corresponds to an <u>elliptically polarized</u> wave!

<u>General case</u> E_1 and E_2 are complex constants

write $E_1 \hat{e}_1 + E_2 \hat{e}_2 \equiv \vec{u}\, e^{i\psi}$
where ψ is chosen so that $\vec{u} \cdot \vec{u}$ is <u>real</u>
– one can always do this since $\vec{u} \cdot \vec{u} = (E_1^2 + E_2^2)\, e^{-2i\psi}$
so 2ψ is just the phase of the complex $E_1^2 + E_2^2$

$\vec{u}$ is a complex vector $\Rightarrow$ $\vec{u} = \vec{u}_a + i\, \vec{u}_b$
with $\vec{u}_a$ and $\vec{u}_b$ real vectors
Since $\vec{u} \cdot \vec{u}$ is real $\Rightarrow$ $\vec{u}_a \cdot \vec{u}_b = 0$
so $\vec{u}_a \perp \vec{u}_b$ orthogonal

Let $\hat{e}_a$ be the unit vector in direction of $\vec{u}_a$
so $\vec{u}_a = u_a \hat{e}_a$ with $u_a = |\vec{u}_a|$
Let $\hat{e}_b = \hat{n} \times \hat{e}_a$ so that $\{\hat{n}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system



Then $\vec{u}_b = \pm\, u_b \hat{e}_b$ where $u_b = |\vec{u}_b|$
since $\vec{u}_b \perp \vec{u}_a$ and both are $\perp$ to $\hat{n}$.

It is $(+)$ if $\vec{u}_b$ is parallel to $\hat{e}_b$ and it is $(-)$ if $\vec{u}_b$ is antiparallel to $\hat{e}_b$.

In this representation we have

$$\vec{E}(\vec{r},t) = Re\left\{ \vec{u}\, e^{i\psi} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$$

$$= e^{-k_2\hat{n}\cdot\vec{r}}\, Re\left\{ u_a \hat{e}_a e^{i(k_1\hat{n}\cdot\vec{r}-\omega t+\psi)} \pm u_b \hat{e}_b (+i) e^{i(k_1\hat{n}\cdot\vec{r}-\omega t+\psi)} \right\}$$

$$= e^{-k_2\hat{n}\cdot\vec{r}} \left\{ u_a \hat{e}_a \cos(\Phi+\psi) \mp u_b \hat{e}_b \sin(\Phi+\psi) \right\}$$

where we write $\Phi \equiv k_1 \hat{n}\cdot\vec{r} - \omega t$

Let's define
$$e^{-k_2\hat{n}\cdot\vec{r}} u_a \rightarrow u_a$$
$$e^{-k_2\hat{n}\cdot\vec{r}} u_b \rightarrow u_b$$
so we don't have to keep writing the constant attenuation factor that is a common factor of all components of $\vec{E}$.

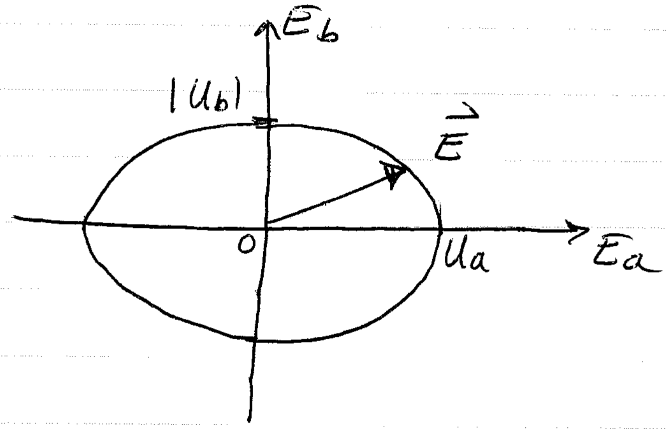
Then define E_a and E_b as the components of $\vec{E}$ in the directions $\hat{e}_a$ and $\hat{e}_b$ respectively.

$$E_a = u_a \cos(\Phi+\psi)$$
$$E_b = \mp u_b \sin(\Phi+\psi)$$

This then gives

$$\left(\frac{E_a}{u_a}\right)^2 + \left(\frac{E_b}{u_b}\right)^2 = \cos^2(\Phi+\psi) + \sin^2(\Phi+\psi) = 1$$

This is just the equation for an ellipse with semiaxes of lengths U_a and U_b, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.



$\Rightarrow$ At a fixed position $\vec{r}$, the tip of the vector $\vec{E}$ will trace out the above ellipse as the time increases by one period of oscillation $2\pi/\omega$.

For (+), ie $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>counterclockwise</u> as t increases

For (−), ie $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>clockwise</u> as t increases

Such a wave is said to be <u>elliptically polarized</u>

<u>Special cases</u>

① $U_a = 0$ or $U_b = 0$
the wave is <u>linearly polarized</u>

(2) $u_a = u_b$

The tip of $\vec{E}$ traces out a ~~[illegible]~~ circle as t increases. The wave is <u>circularly polarized</u>.

The (+) case is said to have <u>right handed circular polarization</u>

The (−) case is said to have <u>left handed circular polarization</u>

One can define circular polarization basis vectors

$$\hat{e}_+ \equiv \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \qquad \hat{e}_- \equiv \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}$$

with $\hat{e}_a$ and $\hat{e}_b$ orthogonal.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_+$ is right handed circularly polarized.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_-$ is left handed circularly polarized.

Just as the general case can always be written as a supperposition of two orthogonal linearly polarized waves, ie

$$\vec{E}_\omega = E_1\,\hat{e}_1 + E_2\,\hat{e}_2$$

One can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\vec{U} = \vec{U}_a + i\vec{U}_b = U_a \hat{e}_a \pm i U_b \hat{e}_b$$

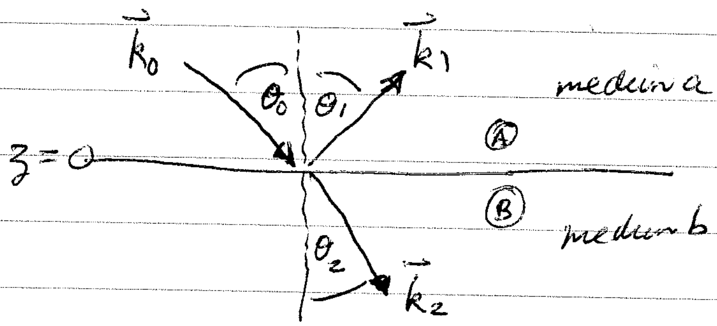
$$= \left(\frac{U_a + U_b}{\sqrt{2}}\right)\hat{e}_{\pm} + \left(\frac{U_a - U_b}{\sqrt{2}}\right)\hat{e}_{\mp}$$

(~~reexpand~~ substitute in for $\hat{e}_{\pm}$ and expand, to see that this is so)

$\Rightarrow$ An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $U_a = 0$ or $U_b = 0$) a <u>linearly</u> polarized wave can always be written as a superposition of circularly polarized waves.

Reflection + Transmission of waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n\, e^{i(\vec{k}_n\cdot\vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or H_n for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \quad \hat{t}\cdot\vec{E}_0\, e^{-i\omega_0 t} + \hat{t}\cdot\vec{E}_1\, e^{-i\omega_1 t} = \hat{t}\cdot\vec{E}_2\, e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t}\cdot\vec{E}_0\, e^{i\vec{k}_0\cdot\vec{\rho}} + \hat{t}\cdot E_1\, e^{i\vec{k}_1\cdot\vec{\rho}} = \hat{t}\cdot\vec{E}_2\, e^{i\vec{k}_2\cdot\vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components $\vec{k}$ vectors can be different

choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $\vec{k}_0$ and $\vec{k}_1$ are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0|\sin\theta_0 = |\vec{k}_1|\sin\theta_1$$

since $k_0^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$ and $k_1^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

<u>angle of incidence = angle of reflection</u>

If ϵ_b is also real and positive (B is transparent) then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin\theta_0 = \sqrt{\mu_b \epsilon_b} \sin\theta_2$$

in terms of index of refraction $n = \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c} \frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law
true for all types of waves, not just EM waves

If $n_a > n_b$ then $\theta_2 > \theta_0$
In this case, when θ_0 is too large, we will have $\frac{n_a}{n_b}\sin\theta_0 > 1$ and there will be no solution for θ_2
$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

$$\frac{n_a}{n_b}\sin\theta_c = 1, \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$