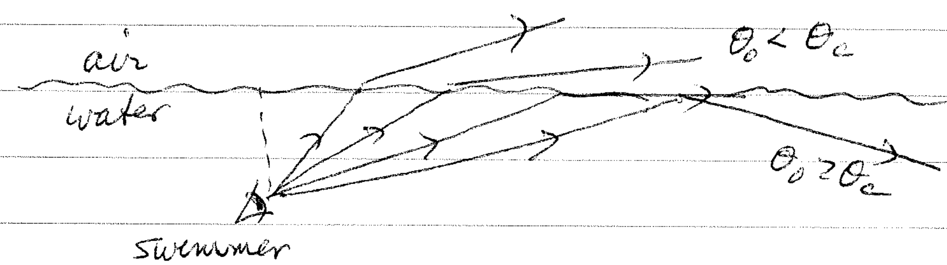
$\epsilon \simeq 1 + 4\pi N \alpha$ ← density

Since $n \approx \sqrt{\mu \epsilon}$ and ϵ grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large $n \Rightarrow$ small θ_c $\Rightarrow$ light bounces around inside many times before it can exit

Can also see total internal reflection when swimming under water



More general case $\sqrt{\epsilon_b}$ is complex so $\vec{k}_2$ is complex

$$\vec{k}_2 = \vec{k}_2' + i\vec{k}_2'' \qquad k_2' = |\vec{k}_2'|$$

$$k_2'' = |\vec{k}_2''|$$

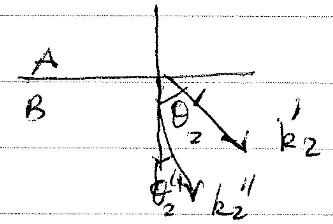
↑ real part ↑ imaginary part

Note $\vec{k}_2'$ and $\vec{k}_2''$ need not be in the same direction!

Condition $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' \\ 0 = k_{2x}'' \end{cases}$ equate real and imaginary parts

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

$$0 = k_2'' \sin\theta_2''$$



$\Rightarrow$ $\theta_2'' = 0$, $\vec{k}_2'' = k_2'' \hat{z}$ } attenuation factor for the transmitted wave is $e^{-k_2 z}$

$\Rightarrow$ planes of constant amplitude are always parallel to the interface nomatter what the angle of incidence θ_0

Having found θ_2'' there are still three quantities we must yet find in order to characterize the transmitted wave. These are θ_2', k_2', k_2''.

To solve for these we will need 3 equations

one is: $\quad k_0 \sin\theta_0 = k_2' \sin\theta_2' \quad$ (1)

(from boundary condition)

where $k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$ $\quad$ dispersion relation in medium a

The other two come from equating the real and imaginary parts of the dispersion relation in medium b.

$$k^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_b = \frac{\omega^2}{c^2}\mu_b(\epsilon_{b1} + i\epsilon_{b2})$$

$$\begin{aligned} k^2 &= (\vec{k}_2' + i\vec{k}_2'')\cdot(\vec{k}_2' + i\vec{k}_2'') \\ &= (k_2')^2 - (k_2'')^2 + 2i\,\vec{k}_2'\cdot\vec{k}_2'' \\ &= (k_2')^2 - (k_2'')^2 + 2i\,k_2' k_2'' \cos\theta_2' \end{aligned}$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad (2)$$

$$2 k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b2} \qquad (3)$$

Use (2) and (3) to solve for k_2' and k_2'' in terms of θ_2'

$$(2) \Rightarrow (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad (4)$$

$$(3) \Rightarrow k_2'' = \frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2' \cos\theta_2'} \qquad (5)$$

plug (5) into (4)

$$(k_2')^2 = \left(\frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2' \cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1}$$

$$\Rightarrow (k_2')^4 - \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2 \varepsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

solve quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \varepsilon_{b1}}{2c^2} + \sqrt{\frac{\omega^4 \mu_b^2 \varepsilon_{b1}^2}{4c} + \frac{\omega^4 \mu_b^2 \varepsilon_{b2}^2}{4c^4\cos^2\theta_2'}}$$

take (+) solution only since $(k_2')^2$ must be positive

$$= \frac{\omega^2 \mu_b}{c^2}\left[\frac{\varepsilon_{b1}}{2} + \frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_2'}}\right]$$

(6) $$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

then get k_2'' from (4)

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b\epsilon_{b1}$$

(7) $$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

Note, these reduce to what we found earlier for the real and imaginary parts of the wave vector for a plane wave in a medium with complex ε, IF we take $\theta_2' = 0$. We will have $\theta_2'' = 0$ for normal incidence $\theta_0 = 0$.

Both k_2' and k_2'' above still depend on the angle of refraction θ_2'. We can close the set of equation by adding in Eq (1)

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

(8) or $$\frac{\omega}{c} n_a \sin\theta_0 = k_2' \sin\theta_2'$$

where $n_a = \frac{k_0 c}{\omega} = \sqrt{\mu_a \varepsilon_a}$

Since the pair of equations (6) and (8) only involve the unknowns k_1' and θ_2' we can use them to eliminate k_2' and get a final single equation that determines θ_2')

Define index of refraction in medium b

$$n_b = \sqrt{\mu_b \varepsilon_{b1}}$$

Then

$$\frac{\omega}{c} n_a \sin\theta_0 = \frac{\omega}{c} n_b \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2} \sin\theta_2'$$

or

$$\boxed{n_a \sin\theta_0 = n_b \sin\theta_2' \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2}}$$

This is the analog of Snell's law for propagation into a medium with complex dielectric function ε

<u>Cases</u>

① For a nearly transparent material with $\varepsilon_{b2} \ll \varepsilon_{b1}$ we can expand in $\frac{\varepsilon_{b2}}{\varepsilon_{b1}}$ to get

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'}\right]^{1/2}$$

$$\simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'}\right]$$

small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b}\sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]$$

or $$\sin\theta_2' \simeq \frac{n_a}{n_b}\sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]}$$

$$\leq \frac{n_a}{n_b}\sin\theta_0$$

result is that θ_2' is <u>smaller</u> than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very different from Snell's Law!

Snell's Law only holds if both media are transparent

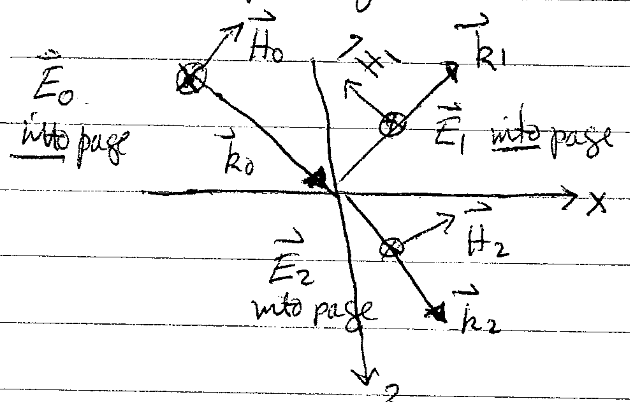
Reflection coefficients

Now we compute the amplitude of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

Consider two cases ① $\vec{E}_\omega$ is $\perp$ plane of incidence
② $\vec{E}_\omega$ lies in the plane of incidence

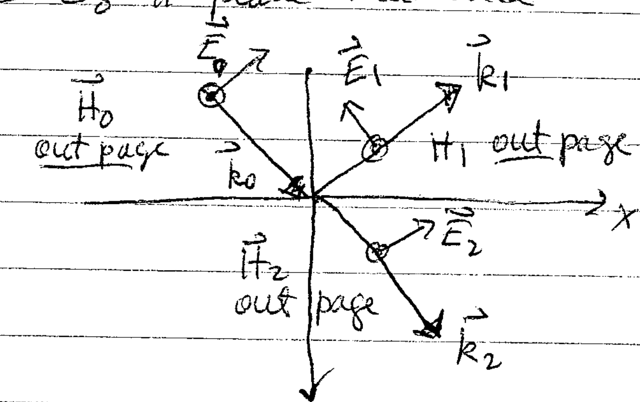
"plane of incidence" is the plane spanned by the wave vector $\vec{k}_0$ and the normal to the interface – in our case it is the xz plane

① $\vec{E}_0 \perp$ plane of incidence



$\Rightarrow \vec{H}_0$ in plane of incidence
all $\vec{E}$'s are in $\hat{y}$ direction

② $\vec{E}_0 \parallel$ plane incidence



$\Rightarrow \vec{H}_0 \perp$ plane of incidence
all the $\vec{H}$'s are in y direction

Continuity of y components

1) $E_0 + E_1 = E_2$ | 1) $H_0 + H_1 = H_2$

continuity of x components

$$H_{0x} + H_{1x} = H_{2x}$$

Faraday

$$\frac{i\mu\omega}{c}\vec{H} = i\vec{k}\times\vec{E} \Rightarrow H_x = \frac{k_z c}{\omega\mu}E_y$$

$$E_{0x} + E_{1x} = E_{2x}$$

Ampere

$$-\frac{i\omega\epsilon}{c}\vec{E} = i\vec{k}\times\vec{H} \Rightarrow E_x = \frac{-k_z c}{\omega\epsilon}H_y$$

$\Rightarrow$

2) $\dfrac{k_{0z}}{\mu_a}(E_0 - E_1) = \dfrac{k_{2z}}{\mu_b} E_2$

solve (1) and (2) for E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$$

$$E_2 = \frac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$$

2) $\dfrac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \dfrac{k_{2z}}{\epsilon_b} H_2$

solve (1) and (2) for H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$$

$$H_2 = \frac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$$

Define reflection coefficient in terms of the transported energy $\quad R = \dfrac{|E_1|^2}{|E_0|^2} = \dfrac{|H_1|^2}{|H_0|^2}$

<u>Reflection coefficients</u>

① $\vec{E}_0 \perp$ plane incidence

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left|\frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}}\right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left|\frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}}\right|^2$$

Note: above are correct for an <u>arbitrary</u> medium B

i) Consider region of "total reflection"

$$\Rightarrow \quad \left.\begin{aligned} \operatorname{Im}\epsilon_b &= \epsilon_{b2} \approx 0 \\ \operatorname{Re}\epsilon_b &= \epsilon_{b1} < 0 \end{aligned}\right\} \Rightarrow \vec{k}_2 = i\vec{K}_2 \quad \text{where } \vec{K}_2 \text{ is real}$$

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left|\frac{a - ib}{a + ib}\right|^2 = 1$ when a, b real

$$\Rightarrow R_\perp = R_\parallel = 1$$

confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$

$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

can write $R_\perp$ and $R_\parallel$ as functions of θ_0

for simplicity take $\mu_a = \mu_b = 1$

(1) $$R_\perp = \left(\frac{n_a\cos\theta_0 - n_b\cos\theta_2}{n_a\cos\theta_0 + n_b\cos\theta_2}\right)^2 = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_2\cos\theta_0 - \sin\theta_0\cos\theta_2}{\sin\theta_2\cos\theta_0 + \sin\theta_0\cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$ if $n_a = n_b$, no reflection! (not surprising!)

(2) $$R_{\parallel} = \left(\frac{\epsilon_b n_a\cos\theta_0 - \epsilon_a n_b\cos\theta_2}{\epsilon_b n_a\cos\theta_0 + \epsilon_a n_b\cos\theta_2}\right)^2$$ use $\sqrt{\epsilon_b} = n_b$, $\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b\cos\theta_0 - n_a\cos\theta_2}{n_b\cos\theta_0 + n_a\cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0\cos\theta_0 - \sin\theta_2\cos\theta_2}{\sin\theta_0\cos\theta_0 + \sin\theta_2\cos\theta_2}\right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}\right)^2$$ ← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So <u>for $\theta_0 = 0$</u>, $R_{\parallel} = R_{\perp}$ – this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as <u>Brewster's angle</u> θ_B, determined by

$$\underbrace{n_a \sin \theta_B}_{\theta_0} = \underbrace{n_b \sin\left(\tfrac{\pi}{2} - \theta_B\right)}_{\theta_2} = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at θ_B, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is <u>always</u> linearly polarized with $\vec{E}_1 \perp$ plane of incidence. – This is one method to create polarized light wave.