Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega)\,\vec{E}_\omega$

Causal response ie $\tilde{\alpha}(t) = 0$ for $t < 0$
$\Rightarrow$ $\alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \oint \frac{\alpha(\omega')}{\omega' - \bar{\omega}}\, d\omega'$$ since no poles of α in UHP

contour along real axis, closed at infinity in UHP. The closing ~~loop~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \to \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\alpha(\omega')}{\omega' - \bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \to 0$

$$\alpha(\omega) = \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\alpha(\omega')}{\omega' - \omega - i\delta}$$

Now $$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\,\delta(\omega' - \omega)$$

principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(\omega')\,d\omega'}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\alpha(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\alpha(\omega')\,d\omega'}{\omega' - \omega} \\ \mathrm{Im}\,\alpha(\omega) &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\alpha(\omega')\,d\omega'}{\omega' - \omega} \end{aligned}}$$

Kramer Kronig relations

If know $\mathrm{Re}\,\alpha$ <u>or</u> $\mathrm{Im}\,\alpha$ can reconstruct full complex α

True for any causal response function

Radiation From Moving Charges

In Lorentz gauge $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$$\left.\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{j} \end{aligned}\right\} \text{ wave equation with source}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics: $\nabla^2\phi = -4\pi\rho$

Greens function satisfies $\nabla^2 G(\vec{r}) = -4\pi\delta(\vec{r})$

then $\phi(\vec{r}) = \int d^3r' \, G(\vec{r}-\vec{r}')\rho(\vec{r}') + \phi_0$

solution for infinite volume that vanishes as $r\to\infty$ is

$$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \qquad \nabla^2\phi_0 = 0$$

For wave equation we want solution to

$$\nabla^2 G(\vec{r},t;\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r}t;\vec{r}'t')}{\partial t^2} = -4\pi\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

Then we will have
$$\begin{cases} \phi(\vec{r},t) = \int dt' \int d^3r' \, G(\vec{r},t;\vec{r}'t')\,\rho(\vec{r}',t') + \phi_0 \\ \vec{A}(\vec{r},t) = \frac{1}{c}\int dt' \int d^3r' \, G(\vec{r}t;\vec{r}'t')\,\vec{j}(\vec{r}',t') + \vec{A}_0 \end{cases}$$

where $\nabla^2\phi_0 - \frac{1}{c^2}\frac{\partial^2\phi_0}{\partial t^2} = 0$ similarly for $\vec{A}_0$

ϕ_0 and $\vec{A}_0$ could describe an incoming wave for example

To construct the Green's function.

For <u>infinite space</u> (but not, for example, inside a cavity)

$G(\vec{r}t; \vec{r}'t') = G(\vec{r}-\vec{r}', t-t')$

express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\| \qquad = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$-4\pi\delta(\vec{r})\delta(t) = -4\pi \int \frac{d^3k\, d\omega}{(2\pi)^4} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

equate Fourier amplitudes

$$\Rightarrow \quad \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\boxed{\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2c^2 - \omega^2}} \qquad \text{when } \omega^2 \neq c^2k^2$$

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2c^2 - \omega^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$\uparrow$ poles at $\omega = \pm ck$

In evaluating the ω integral we have to know how to treat the poles on the real axis so that $G(\vec{r},t)$ will have the desired behavior.

What we want is for $G(\vec{r},t)$ to be <u>causal</u>, ie $G(\vec{r},t)=0$ for $t<0$, so $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend only on the values of the sources at <u>earlier</u> times $t'<t$.

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},\omega) = 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2\, e^{ikr\cos\theta}\,\tilde{G}(k,\omega)$$

$\mu \equiv \cos\theta$

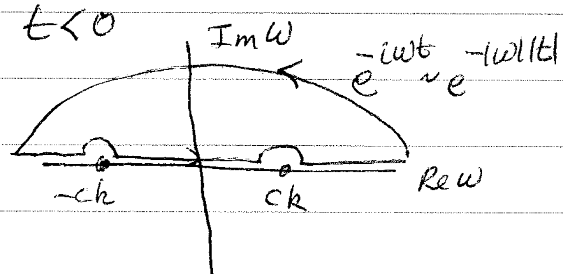
$$= 2\pi \int_{-1}^{1} d\mu \int_0^{\infty} dk\, k^2\, e^{ikr\mu}\,\tilde{G}(k,\omega)$$

$$= 4\pi \int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\,\tilde{G}(k,\omega)$$

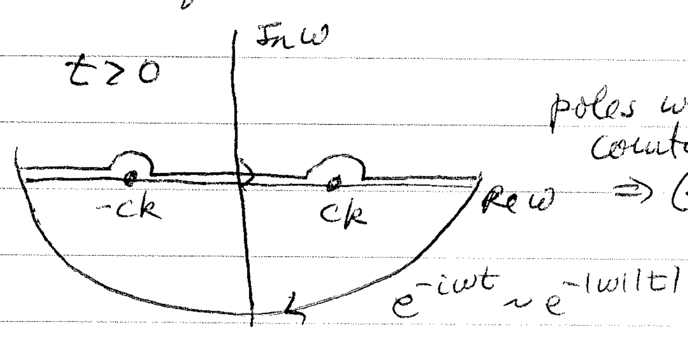
$$G(\vec{r},t) = -\frac{c^2}{\pi^2}\int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\int_C \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)}\,d\omega$$

contour along real axis, but deformed to go around the poles

for $t<0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the <u>upper half complex</u> (UHP) ω plane $\Rightarrow$ can close contour in UHP for $t<0$. If we want $G=0$ for $t<0$, there should therefore be no poles in UHP. The contour C we want is therefore:



no poles in contour $\Rightarrow G=0$



poles within contour $\Rightarrow G\neq 0$

with this convention for the contour C we can evaluate the ω-integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}\,d\omega}{(\omega+ck)(\omega-ck)} = -2\pi i\left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck}\right]$$

$$= -\frac{2\pi\sin(ckt)}{ck}$$

$$G(\vec{r},t) = \frac{2c}{\pi r}\int_0^\infty dk\,\sin(kr)\sin(ckt) = \frac{c}{\pi r}\int_{-\infty}^{\infty} dk\,\frac{(e^{ikr}-e^{-ikr})(e^{ickt}-e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k}\right]$$

each integral would give a δ-function, but for 1st two terms $\delta(r+ct)=0$ since here $t>0$ (by definition) and $r=|\vec{r}|\geq 0$ so the arguement will never vanish.

$$\boxed{G(\vec{r},t) = \frac{c}{r}\,\delta(r-ct) = \frac{\delta(t-r/c)}{r}}$$

using $\delta(ax)=\frac{1}{a}\delta(x)$

$$G(\vec{r}t,\vec{r}'t') = \begin{cases} \dfrac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} & t-t'>0 \\ 0 & t-t'<0 \end{cases}$$

Green's function for wave equation in free space

$G\neq 0$ only on "light cone" that ~~connects~~ emanates from (r',t'), ie when $|\vec{r}-\vec{r}'| = c(t-t')$.

Signal from source at (r',t') travels with c

$$\phi(\vec{r},t) = \phi_0(\vec{r},t) + \int d^3r' \int_{-\infty}^{t} dt' \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \rho(\vec{r}',t')$$

$$A(\vec{r},t) = \vec{A}_0(\vec{r},t) + \int \frac{d^3r'}{c} \int_{-\infty} dt' \frac{\delta\left(t-t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}',t')$$

Apply to a <u>single moving point charge</u>

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t)) \qquad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r},t) = q \int dt' \frac{\delta(t-t' - \frac{1}{c}|\vec{r}-\vec{r}_0(t')|)}{|\vec{r}-\vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t'-t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c}|\vec{r}-\vec{r}_0(t')|$$

then

$$\phi(\vec{r},t) = q \int dt' \frac{\delta(t-g(t'))}{|\vec{r}-\vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t-g(t'))}{|\vec{r}-\vec{r}_0(t')|}\, dg \left(\frac{dt'}{dg}\right)$$

$$= \frac{q}{|\vec{r}-\vec{r}_0(t')|} \frac{1}{(dg/dt')}\Bigg|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c}\sqrt{[x - x_0(t')]^2 + [y - y_0(t')]^2 + [z - z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c\,|r - r_0(t')|}\left\{[x - x_0(t')]\left(-\frac{dx_0}{dt'}\right) + \cdots\right\}$$

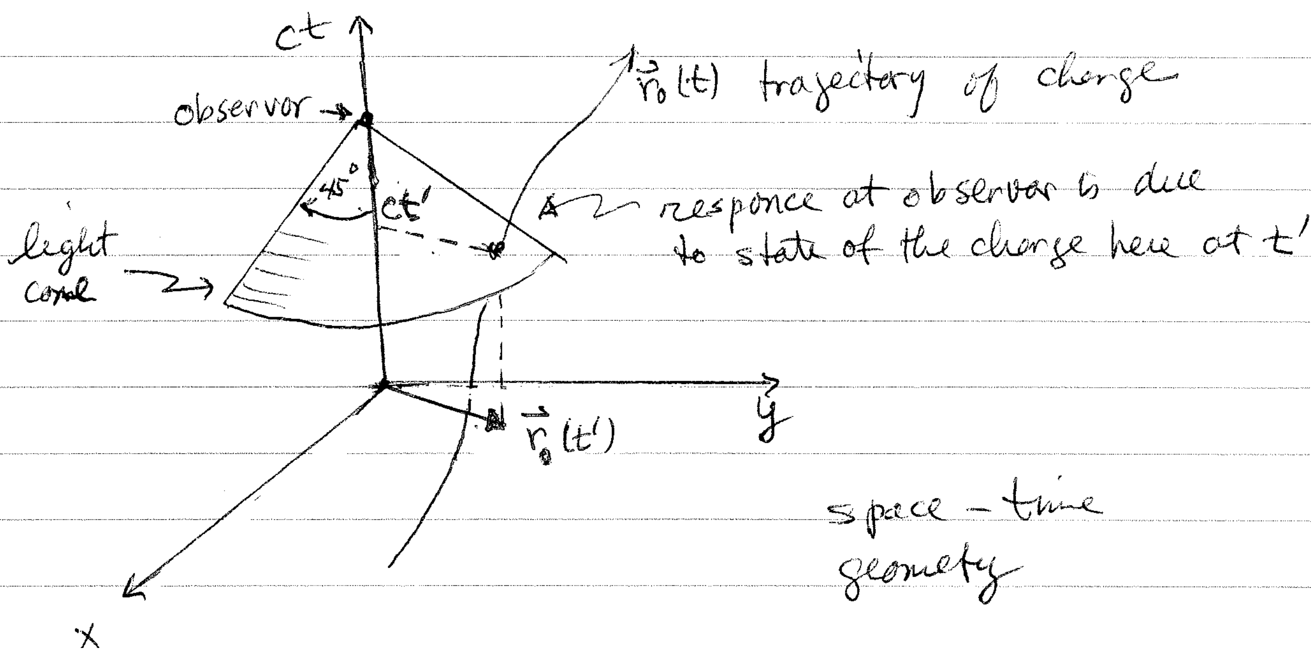
$$= 1 - \frac{1}{c}\,\hat{n}(t')\cdot\vec{v}(t')$$

where $\hat{n}(t') = \dfrac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$ unit vector pointing from $\vec{r}_0(t')$ to $\vec{r}$

$$\left.\begin{aligned}
\phi(\vec{r}, t) &= \frac{q}{|\vec{r} - \vec{r}_0(t')|\left[1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')\right]} \\[2ex]
\vec{A}(\vec{r}, t) &= \frac{q\,\vec{v}(t')/c}{|\vec{r} - \vec{r}_0(t')|\left[1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')\right]}
\end{aligned}\right\}\ \text{Lienard - Wiechert Potentials}$$

where t' is determined by the condition

$$t - t' = \frac{1}{c}|\vec{r} - \vec{r}_0(t')|$$



space - time geometry