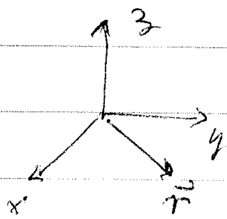
For charge moving with constant velocity along $\hat{z}$

$$\vec{r}_0(t) = vt\,\hat{z} \qquad \vec{v} = \frac{d\vec{r}_0}{dt} = v\,\hat{z}$$

For observer at position $\vec{r}$ (in xy plane), time $t$, the fields will be determined by the charge at time $t'$ such that




$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$

$$(t - t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2)\,t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

let $\gamma = (1 - v^2/c^2)^{-1/2}$

$$t'^2 - 2\gamma^2 t\,t' + \gamma^2 \frac{(c^2 t^2 - r^2)}{c^2} = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2\left(\gamma^2 t^2 - t^2 + \frac{r^2}{c^2}\right)}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left(\frac{v^2}{c^2}\right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma^2}{c} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

consider $t=0$. Solution should give $t'<0$

$\Rightarrow$ $(-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c}\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

$$\phi(\vec{r},t) = \frac{q}{|\vec{r}-\vec{r}_0(t')|\left[1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}\right]}$$

$$|\vec{r}-\vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t-t')$$ from condition that determines $t'$

$$(\vec{r}-\vec{r}_0(t'))\cdot\vec{v} = -\vec{r}_0(t')\cdot\vec{v}$$ for $\vec{v} = v\hat{z}$

$$= -v^2 t'$$ $\vec{r}$ in xy plane

$$\phi(\vec{r},t) = \frac{q}{c(t-t')\left[1 + \frac{v^2 t'}{c^2(t-t')}\right]}$$

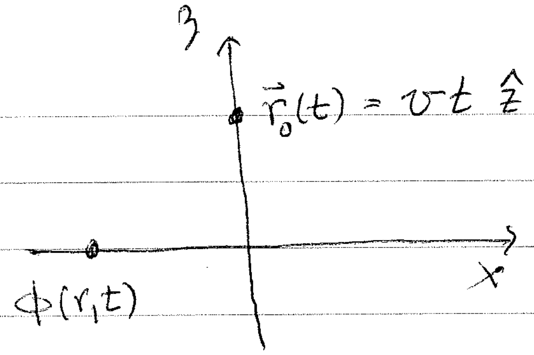
$$= \frac{q}{c(t-t') + \frac{v^2 t'}{c}} = \frac{q}{c\left[t - \left(1 - \frac{v^2}{c^2}\right)t'\right]}$$

$$= \frac{q}{c\left(t - \frac{t'}{\gamma^2}\right)} = \frac{q}{c\,\frac{1}{c}\sqrt{v^2 t^2 + r^2/\gamma^2}}$$

$$\phi(\vec{r},t) = \frac{q}{\sqrt{v^2 t^2 + r^2/\gamma^2}}$$

$$\vec{A}(\vec{r},t) = \frac{q\vec{v}}{c\sqrt{v^2 t^2 + r^2/\gamma^2}}$$

$\Big\}$ solutions for $\vec{r}$ in xy plane when charge passes through xy plane at $t=0$




at $x$
~~potential from q at $vt\,\hat{z}$~~

potential at pt $\vec{r}$ in $xy$ plane at time $t$, when charge is at $\vec{r}_0 = vt\,\hat{z}$, looks almost like static Coulomb potential, which would be $\dfrac{q}{\sqrt{r^2+v^2t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2t^2+\left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted by a factor $\gamma$!

Such considerations led Lorentz to discover the Lorentz transformation, <u>before</u> Einstein proposed his theory of special relativity

# Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r'\,dt'\ \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\ \vec{j}(\vec{r}',t')$$

For pure harmonic oscillation in current

$$\vec{j}(\vec{r},t) = Re\left\{\vec{j}_\omega(\vec{r})\,e^{-i\omega t}\right\}$$

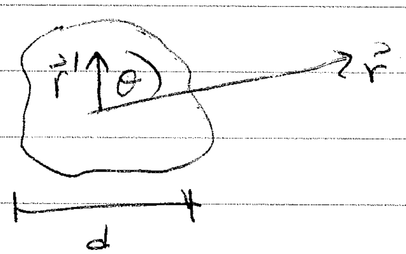
$$\Rightarrow\ \vec{A}(r,t) = Re\left\{\vec{A}_\omega(\vec{r})\,e^{-i\omega t}\right\}$$

$$\Rightarrow\ \vec{A}_\omega(\vec{r})e^{-i\omega t} = \frac{1}{c}\int d^3r'\ \vec{j}_\omega(\vec{r}')\,e^{-i\omega t}\,\frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

doing $\int dt'$ by using the $\delta$-function

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r'\ \vec{j}_\omega(\vec{r}')\,\frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume source is localized, ie $\vec{j}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$




Approx ①

for $r \gg d$, far from sources

$$|\vec{r}-\vec{r}'| = \sqrt{r^2+r'^2-2rr'\cos\theta}$$

$$= r\sqrt{1+\left(\frac{r'}{r}\right)^2-\frac{2r'}{r}\cos\theta}$$

$$\simeq r\left(1-\frac{r'}{r}\cos\theta\right)$$

$$\simeq r-\vec{r}'\cdot\hat{r} + o\left(\frac{r'}{r}\right)^2$$

$$\hat{r} \equiv \frac{\vec{r}}{r}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}') \, e^{ik(r-\vec{r}'\cdot\hat{r})}}{r-\vec{r}'\cdot\hat{r}} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}') \, e^{-ik\vec{r}'\cdot\hat{r}}}{1-\frac{\hat{r}\cdot\vec{r}'}{r}}$$

$$\approx \frac{e^{ikr}}{cr}\int d^3r' \, \vec{J}_\omega(\vec{r}') \, e^{-ik\hat{r}\cdot\vec{r}'}\left(1+\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \, \vec{J}_\omega(\vec{r}') \cdots$ term will determine the angular dependence of the radiation.

Approx 2 $\quad \lambda \gg d \quad$ long wave length approx

or $kd \ll 1 \Rightarrow \frac{\omega}{c} d \ll 1$ or $\frac{d}{\tau} \ll c$

where $\tau$ is period of oscillation.

Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\Rightarrow \lambda \gg d$ is a non-relativistic approximation

$$kd \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}'} \simeq 1 - ik\hat{r}\cdot\vec{r}' + \text{higher orders}$$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \left(1 - ik\hat{r}\cdot\vec{r}'\right)\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \left[1 + \hat{r}\cdot\vec{r}'\left(\frac{1}{r} - ik\right)\right]$$

+ higher order in $\frac{d}{r}$ or $kd$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r}\left[\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right]$$

where $\vec{I}_1 \equiv \frac{1}{c}\int d^3r' \, \vec{j}_\omega(\vec{r}')$

$\vec{I}_2 \equiv \frac{1}{c}\int d^3r' \, \hat{r}\cdot\vec{r}' \, \vec{j}_\omega(\vec{r}')$

Consider first $I_1$ $i$th component ($\vec{I}_1$ vanishes in statics)

$$\int d^3r \, j_i(\vec{r}) = -\int d^3r \, r_i \vec{\nabla}\cdot\vec{j} \qquad \text{integration by parts}$$

$$= \int d^3r \, r_i \frac{\partial \rho}{\partial t} \qquad \text{as } \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0$$

$$\int d^3r \, j_i(\vec{r}) = -i\omega \int d^3r \, r_i \rho_\omega(\vec{r})$$

$$\Rightarrow \vec{I}_1 = -\frac{i\omega}{c}\int d^3r \, \vec{r}\rho_\omega(\vec{r}) = -\frac{i\omega}{c}\vec{p}_\omega$$

electric dipole moment

Electric dipole approximation from $I_1$

$$\boxed{\vec{A}_{E1}(\vec{r}) = \frac{e^{ikr}}{r}\left(\frac{-i\omega}{c}\vec{p}_\omega\right) = -i\,\vec{p}_\omega \frac{k e^{ikr}}{r}}$$ $\omega = ck$

Consider $\vec{I}_2$

$$\vec{I}_2 = \frac{1}{c}\int d^3r'\ \hat{r}\cdot\vec{r}'\,\vec{j}_\omega(\vec{r}') = \frac{1}{c}\hat{r}\cdot\int d^3r'\ \underbrace{\vec{r}'\,\vec{j}_\omega(\vec{r}')}_{\text{tensor}}$$

we saw this tensor earlier when we did the magnetic dipole approx, and when we derived the macroscopic Maxwell equations

$$\int d^3\vec{r}'\ \vec{r}'\vec{j}_\omega(\vec{r}') = -\int d^3r'\ \vec{j}_\omega(\vec{r}')\,\vec{r}' - \int d^3r'\ \vec{r}'\vec{r}'(\vec{\nabla}'\cdot\vec{j}_\omega(r'))$$

$$= \frac{1}{2}\int d^3r'\left[\vec{r}'\vec{j}_\omega - \vec{j}_\omega\vec{r}'\right] - \frac{1}{2}\int d^3r'\ i\omega\,\vec{r}'\vec{r}'\rho_\omega$$

using $\vec{\nabla}'\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$$\vec{I}_2 = \frac{1}{2c}\int d^3r'\left[(\hat{r}\cdot\vec{r}')\vec{j}_\omega - (\hat{r}\cdot\vec{j}_\omega)\vec{r}'\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'(\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\frac{1}{2c}\int d^3r'\left[\hat{r}\times(\vec{r}'\times\vec{j}_\omega)\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'\ (\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}'_\omega$$

where $\vec{m}_\omega = \frac{1}{2c}\int d^3r'\ \vec{r}'\times\vec{j}_\omega(\vec{r}')$ is magnetic dipole moment

$\overleftrightarrow{Q}'_\omega = \int d^3r'\ 3\vec{r}'\vec{r}'\rho_\omega(\vec{r}')$ looks almost like electric quadrapole tensor to make it look like the proper quadrapole moment

$$\overleftrightarrow{Q}_\omega = \int d^3r'\left(3\vec{r}'\vec{r}' - r'^2\overleftrightarrow{I}\right)\rho_\omega(\vec{r}')$$

we can write

$$\overleftrightarrow{Q}'_\omega = \overleftrightarrow{Q}_\omega + \overleftrightarrow{I}\int d^3r'\, r'^2\rho_\omega(\vec{r}')$$

identity matrix $I_{ij} = \delta_{ij}$

$$\overleftrightarrow{I}_2 = -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}_\omega - \frac{i\omega}{6c}\hat{r}\,C(\omega)$$

where $C_\omega \equiv \int d^3r'\, r'^2\rho_\omega(\vec{r}')$ is a scalar

<u>Magnetic dipole approximation from $\overleftrightarrow{I}_2$</u>

$$\boxed{\vec{A}_{M1}(\vec{r}) = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\hat{r}\times\vec{m}_\omega\right)}$$

<u>Electric quadrapole approximation from $\overleftrightarrow{I}_2$</u>

$$\boxed{\vec{A}_{E2}(\vec{r}) = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\frac{i\omega}{6c}\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)}$$

The last piece $\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(\frac{-i\omega}{6c}\hat{r}\,C_\omega\right)$

can always be ignored - it is a radial function and so its curl always vanishes $\Rightarrow$ gives no contribution to $\vec{B}$. Similarly, since $\frac{-i\omega}{c}\vec{E}_\omega = i\vec{k}\times\vec{B}_\omega$ by Ampere's law, this term will give no contribution to $\vec{E}$.

holds away from source where $\vec{J}\approx 0$.