Examples with azimuthal symmetry $m=0$

General solution to $\nabla^2\phi=0$ can be written in form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$$

determine the A_ℓ and B_ℓ from the boundary conditions of the particular problem.

① Suppose one is given $\phi(R,\theta) = \phi_0(\theta)$ on surface of sphere of radius R.

To find solution of $\nabla^2\phi=0$ inside sphere

ϕ should not diverge at origin $\Rightarrow B_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \phi(R,\theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \int_0^\pi d\theta \sin\theta\, \phi_0(\theta) P_m(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_m(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) \delta_{\ell m}$$

$$= A_m R^m \frac{2}{2m+1}$$

$$\boxed{A_m = \frac{2m+1}{2R^m} \int_0^\pi d\theta \sin\theta\, \phi_0(\theta) P_m(\cos\theta)}$$ gives solution

To find solution of $\nabla^2\phi = 0$ <u>outside</u> sphere

if require $\phi \to 0$ as $r \to 0$, then $A_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi(R,\theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

gives solution
$$\boxed{B_m = \frac{2m+1}{2} R^{m+1} \int_0^{\pi} d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta)}$$

$$\boxed{B_m = A_m R^{2m+1}}$$

(2) Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius R. What is ϕ inside ad outside?

From previous example

$$\phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & r > R \end{cases}$$

boundary conditions at $r = R$ on surface

(i) ϕ continuous

$$\to \sum_{\ell=0}^{\infty} \left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}} \right] P_\ell(\cos\theta) = 0$$

If an expansion in Legendre polynomials vanishes for all θ, then each coefficient in the expansion must vanish

$$\Rightarrow \quad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \Rightarrow \quad \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

(ii) jump in electric field at σ

$$-\left.\frac{\partial \phi^{out}}{\partial r}\right|_{r=R} + \left.\frac{\partial \phi^{in}}{\partial r}\right|_{r=R} = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty} \left[\frac{(\ell+1) B_\ell}{R^{\ell+2}} + \ell A_\ell R^{\ell-1} \right] P_\ell(\cos\theta) = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty} \left[\frac{(\ell+1) A_\ell R^{2\ell+1}}{R^{\ell+2}} + \ell A_\ell R^{\ell-1} \right] P_\ell(\cos\theta)$$

$$\Rightarrow \sum_{\ell=0}^{\infty} (2\ell+1) R^{\ell-1} A_\ell P_\ell(\cos\theta) = 4\pi\sigma$$

$$(2m+1) R^{m-1} A_m \left(\frac{2}{2m+1}\right) = 4\pi \int_0^\pi d\theta \sin\theta \, \sigma(\theta) P_m(\cos\theta)$$

$$\boxed{A_m = \frac{4\pi}{2R^{m-1}} \int_0^\pi d\theta \sin\theta \, \sigma(\theta) P_m(\cos\theta)}$$

Suppose $\sigma(\theta) = k\cos\theta$ what is ϕ?

Note $\sigma(\theta) = k P_1(\cos\theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_\ell(\cos\theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi d\theta \sin\theta \, P_1(\cos\theta) P_1(\cos\theta)$$

$$= \frac{4\pi k}{2} \left(\frac{2}{2+1} \right) = \frac{4}{3}\pi k$$

$$\Rightarrow \phi(r,\theta) = \begin{cases} \frac{4\pi}{3} k \, r\cos\theta & r < R \\ \frac{4\pi}{3} k \frac{R^3}{r^2} \cos\theta & r > R \end{cases}$$

we will see that potential outside the sphere is that of an ideal dipole with dipole moment

$$p = \frac{4}{3}\pi R^3 k$$

Inside the sphere, the potential $\phi = \frac{4\pi}{3} k z$ where $z = r\cos\theta$. The electric field inside the sphere is therefore the constant

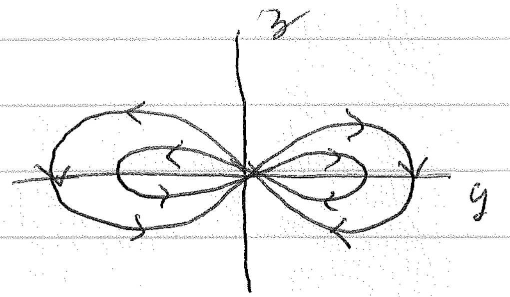
$$\vec{E} = -\vec{\nabla}\phi = -\frac{4}{3}\pi k \hat{z}$$

Outside the sphere the field is

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial \phi}{\partial r}\,\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\,\hat{\theta}$$

$$= \frac{8\pi}{3} k \frac{R^3}{r^3} \cos\theta\,\hat{r} + \frac{4\pi}{3} k \frac{R^3}{r^3} \sin\theta\,\hat{\theta}$$

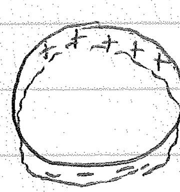
$$\vec{E} = \frac{4\pi R^3 k}{3} \frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



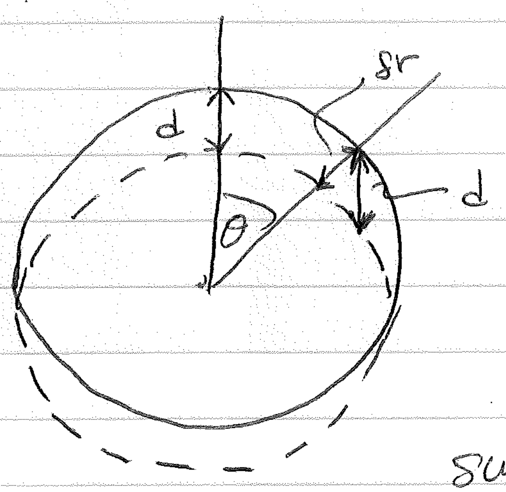
dipole field

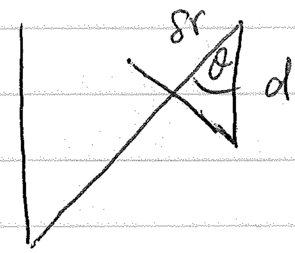
Physical example with $\sigma(\theta) = k\cos\theta$

Two spheres of radii R, with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement
This is a uniformly "polarized" sphere





$d\cos\theta = \delta r$

surface charge is $\sigma(\theta) = \rho\,\delta r$
$= \rho d \cos\theta$

$$\boxed{\sigma(\theta) = \rho d \cos\theta}$$

$k \equiv \rho d \equiv \mathbb{P}$

total dipole moment is $(\rho d)\frac{4}{3}\pi R^3$

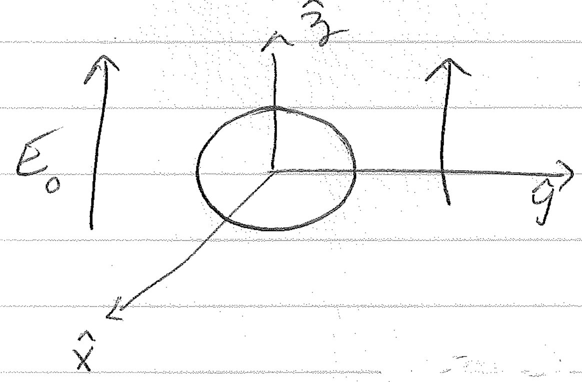
polarization $= \dfrac{\text{dipole moment}}{\text{volume}} = \rho d \equiv \mathbb{P}$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}\hat{z} = -\frac{4\pi \mathbb{P}}{3}\hat{z} = -\frac{4\pi}{3}\vec{\mathbb{P}}$

Grounded

(3) Conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$
$= -E_0 r \cos\theta$



boundary conditions

$$\begin{cases} \phi(R,\theta) = 0 \\ \phi(r\to\infty,\theta) = -E_0 r\cos\theta \end{cases}$$

solution outside sphere has the form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

From boundary condition as $r \to \infty$ we have

$A_\ell = 0$ all $\ell \neq 1$

$A_1 = -E_0$ since $P_1(\cos\theta) = \cos\theta$

$$\phi(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

From $\phi(R,\theta) = 0$ we have

$$0 = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow B_\ell = 0 \text{ all } \ell \neq 1$$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = E_0 R^3$$

So $$\boxed{\phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

1st ten is just potential $-E_0 r\cos\theta$ of the uniform applied electric field.

2nd ten is potential due to the induced surface charge on the surface – it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\frac{\partial\phi}{\partial r}\bigg|_{r=R} = E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta$$

$$= 3E_0\cos\theta$$

$$\sigma(\theta) = \frac{3}{4\pi}E_0\cos\theta$$ like uniformly polarized sphere $k = \frac{3E_0}{4\pi} = P$

from ② we know that the field inside the sphere due to this σ is just $-\frac{4}{3}\pi k\hat{z} = -\frac{4}{3}\pi\frac{3E_0}{4\pi}\hat{z}$

$= -E_0\hat{z}$. This is just what is required so that the total field in the conducting sphere vanishes,

Can check that outside the sphere, $\vec{E} = -\vec{\nabla}\phi$ is normal to surface of sphere at $r = R$.