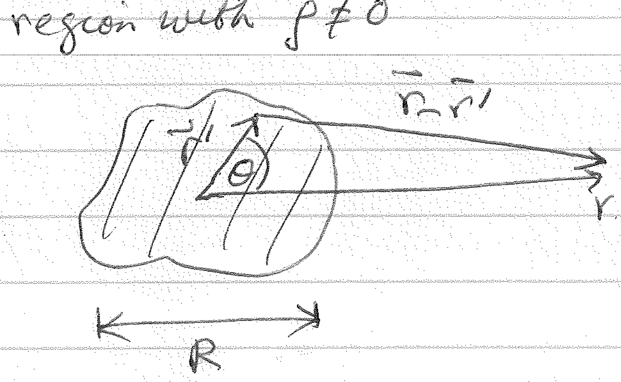
Multipole Expansion

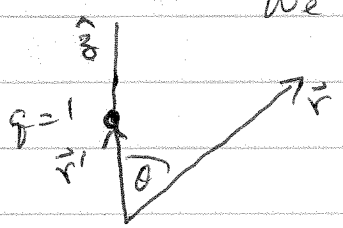


We want to find the potential ϕ for an arbitrary localized distribution of charge ρ, at distances far away $r \gg R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \text{General Coulomb formula}$$

We want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r \gg r'$

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ, so we can express it as an expansion in Legendre polynomials.

For $r > r'$,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad \text{all } A_\ell = 0 \text{ as need } \phi \to 0 \text{ as } r \to \infty$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} P_\ell(\cos\theta)$$

We know $\phi(r, \theta=0) = \dfrac{1}{r-r'}$ (for $r > r'$)

← scalars here since when $\theta = 0$, $\vec{r}$ and $\vec{r}'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r,0) = \frac{1}{r} \sum_{\ell} \frac{B_\ell}{r^\ell} P_\ell(1)$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} \quad \text{as } P_\ell(1) = 1$$

$$= \frac{1}{r} \frac{1}{(1 - r'/r)}$$ ← exact result from Coulomb

Now Taylor expansion $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$

$$\Rightarrow \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots\right)$$

$\Rightarrow B_\ell = (r')^\ell$ is solution

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)}$$

So for the charge distribution ρ,

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \int d^3r' \frac{\rho(\vec{r}')}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \rho(\vec{r}') (r')^\ell P_\ell(\cos\theta)$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r). It is *exact* provided one adds all the infinite l terms. In practice, one generally approximates by summing only up to some finite l.

Note: in doing the integrals

$$\int d^3r' \; \rho(\vec{r}')\,(r')^l\, P_l(\cos\theta)$$

θ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat this integration every time we change $\vec{r}$.

We will find a way around this by
(i) first looking explicitly at the few lowest order terms
(ii) a general method involving spherical harmonics $Y_{lm}(\theta, \varphi)$

monopole: $l=0$ term

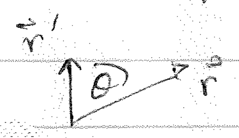
$$\phi^{(0)}(\vec{r}) = \frac{1}{r}\int d^3r'\, \rho(r') \qquad P_0(\cos\theta) = 1$$

$$= \frac{q}{r} \quad \text{where } q = \int d^3r'\, \rho(r') \text{ is}$$

total charge

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\int d^3r'\, \rho(\vec{r}')\, r'\, P_1(\cos\theta)$$



$$= \frac{1}{r^2}\int d^3r'\, \rho(\vec{r}')\, r' \cos\theta$$

Now $\vec{r}\cdot\vec{r}' = rr'\cos\theta \Rightarrow \hat{r}\cdot\vec{r}' = r'\cos\theta$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\,\hat{r}\cdot\int d^3r'\, \rho(\vec{r}')\, \vec{r}'$$

$$= \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \vec{p} \equiv \int d^3r'\, \rho(\vec{r}')\, \vec{r}'$$

is the dipole moment

For a set of point charges q_i at $\vec{r}_i$,

$$\vec{p} = \sum_i q_i \vec{r}_i$$

quadrapole: $\ell = 2$ term

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \int d^3r' \, \rho(\vec{r}') \, r'^2 \, P_2(\cos\theta)$$

$$= \frac{1}{r^3} \int d^3r' \, \rho(\vec{r}') \, r'^2 \, \frac{1}{2}(3\cos^2\theta - 1)$$

use $\cos\theta = \hat{r}' \cdot \hat{r}$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \int d^3r' \, \rho(\vec{r}') \, \frac{1}{2}\left(3(\vec{r}' \cdot \hat{r})^2 - (r')^2\right)$$

$$= \frac{1}{r^3} \, \hat{r} \cdot \left[\int d^3r' \, \rho(\vec{r}') \, \frac{1}{2}\left(3\vec{r}'\vec{r}' - (r')^2 \overleftrightarrow{I}\right) \right] \cdot \hat{r}$$

where $\overleftrightarrow{I}$ is the identity tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u} \cdot \overleftrightarrow{I} \cdot \vec{v} = \vec{u} \cdot \vec{v}$.
and $\vec{r}'\vec{r}'$ is the tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u} \cdot [\vec{r}'\vec{r}'] \cdot \vec{v} = (\vec{u} \cdot \vec{r}')(\vec{r}' \cdot \vec{v})$

Define quadrapole tensor $\overleftrightarrow{Q} \equiv \int d^3r' \rho(\vec{r}') \left(3\vec{r}'\vec{r}' - (r')^2 \overleftrightarrow{I}\right)$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \frac{1}{2} \hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}$$

So to lowest three terms

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{2r^3} + \cdots$$

defined in terms of the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ of the charge distribution.

Note, the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ do not depend on the observation point $\vec{r}$ — we can calculate them once and then use them to get $\phi(\vec{r})$ at all $\vec{r}$.

monopole: $q = \int d^3r\, \rho(\vec{r})$ scalar integral

dipole $\vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r}$ vector integral

$\hat{e}_1 \equiv \hat{x},\ \hat{e}_2 \equiv \hat{y},\ \hat{e}_3 \equiv \hat{z}$

if we pick a coordinate system, we have to do $\underline{\underline{3}}$ integrations to get the three component of $\vec{p}$

$\hat{e}_i \cdot \vec{p} = p_i = \int d^3r\, \rho(\vec{r})\, r_i$

quadrapole $\overleftrightarrow{Q} = \int d^3r\, \rho(\vec{r}) \left(3\vec{r}\,\vec{r} - (r)^2 \overleftrightarrow{I}\right)$ tensor integral

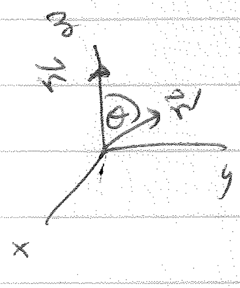
if we pick a coord system $x\ y\ z$ then $\overleftrightarrow{Q}$ is a matrix with components $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

$\hat{e}_i \cdot \overleftrightarrow{Q} \cdot \hat{e}_j = Q_{ij} = \int d^3r\, \rho(\vec{r}) \left[3 r_i r_j - r^2 \delta_{ij}\right]$

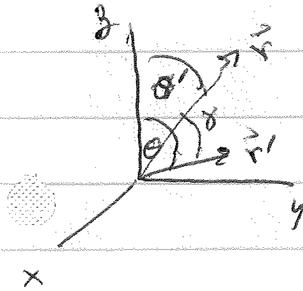
There are 9 elements of the 3×3 matrix Q_{ij}, but $Q_{ij} = Q_{ji}$ is symmetric so there are only 6 independent elements to compute.

General method

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \, \rho(\vec{r}')(r')^\ell \, P_\ell(\cos\theta)$$



in above, θ is angle between $\vec{r}$ and $\vec{r}'$
if we think of this θ as the spherical coord θ, then in effect, above is choosing $\vec{r}$ to be on $\hat{z}$ axis. We would like a representation in which $\vec{r}$ is positioned arbitrarily with respect to the axes used in describing ρ



Use the addition theorem for spherical harmonics
- see Jackson 3.6 for discussion & proof

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi)$$

where (θ, ϕ) are the angles of $\hat{r}$, (θ', ϕ') are the angles of $\hat{r}'$, and γ is the angle between $\hat{r}$ and $\hat{r}'$, ie $\cos\gamma = \hat{r} \cdot \hat{r}'$

$$\cos\theta = \hat{z} \cdot \hat{r}$$
$$\cos\theta' = \hat{z} \cdot \hat{r}'$$

$\Rightarrow$

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \int d^3r' \, \rho(\vec{r}')(r')^\ell \, Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi)$$

Define the moment

$$q_{\ell m} \equiv \int d^3r' \, \rho(\vec{r}')(r')^\ell \, Y^*_{\ell m}(\theta', \phi')$$

independent of observation point

Then

$$\phi(\vec{r}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell m} Y_{\ell m}(\theta,\phi)}{(2\ell+1) r^{\ell+1}}$$

see Jackson equ (4.4), (4-5), (4-6) to relate $q_{\ell m}$ to q, $\vec{p}$, $\overleftrightarrow{Q}$.

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} \ldots$$

electric field $\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$

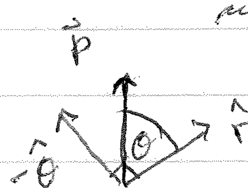
For the monopole term $\vec{E} = \frac{q}{r^2}\hat{r}$

For the dipole term, choose $\vec{p}$ along $\hat{z}$ axis so

$$\phi(\vec{r}) = \frac{p\cos\theta}{r^2}$$

$$\vec{E} = \frac{2p\cos\theta}{r^3}\hat{r} + \frac{p\sin\theta}{r^3}\hat{\theta}$$

$$\vec{E} = \frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



note $p\cos\theta\,\hat{r} = (\vec{p}\cdot\hat{r})\hat{r}$

$p\sin\theta\,\hat{\theta} = -(\vec{p}\cdot\hat{\theta})\hat{\theta}$

Now $\vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$

$\Rightarrow \quad -(\vec{p}\cdot\hat{\theta})\hat{\theta} = (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$

So

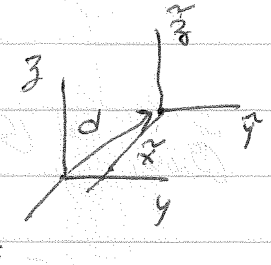
$$\vec{E} = \frac{1}{r^3}\left[2(\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

$$= \frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$ expresses $\vec{E}$ in coord free form

$$\vec{E} = \frac{1}{r^3}\left[\,3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\,\right]$$ expresses $\vec{E}$ of dipole in coord free form

Origin of coordinates

The definition of the multipole moments depends on the choice of origin of the coordinates.



Suppose transform to $\tilde{\vec{r}} = \vec{r} - \vec{d}$

In the $\tilde{\vec{r}}$ cood system

$$\tilde{q} = \int d^3\tilde{r}\ \rho(\tilde{r}) = \int d^3r\ \rho(r) = q$$

monopole does not depend on choice of origin

$$\tilde{\vec{p}} = \int d^3\tilde{r}\ \rho(\tilde{r})\,\tilde{\vec{r}} = \int d^3r\ \rho\,(\vec{r} - \vec{d})$$

$$= \int d^3r\,\rho\,\vec{r} \ - \ \vec{d}\int d^3r\,\rho$$

$$\tilde{\vec{p}} = \vec{p} - \vec{d}q \qquad \tilde{\vec{p}} = \vec{p} \text{ only if } q = 0\,!$$

if $q \neq 0$, then $\tilde{\vec{p}} \neq \vec{p}$

$\Rightarrow$ ~~One coord~~ If $q \neq 0$, one could always choose an origin of coords for which $\vec{p} = 0$!

~~For HW you will show that $\tilde{\overleftrightarrow{Q}} = \overleftrightarrow{Q}$ only if both $q = 0$ and $\vec{p} = 0$~~

Quadrupole moment in new coordinates

$$\tilde{\overleftrightarrow{Q}} = \int d^3\tilde{r}\, \rho \left[3\tilde{\vec{r}}\,\tilde{\vec{r}} - (\tilde{\vec{r}})^2 \overleftrightarrow{I} \right]$$

where $\tilde{\vec{r}} = \vec{r} - \vec{d}$

substitute in above

$$\tilde{\overleftrightarrow{Q}} = \int d^3r\, \rho \left[3(\vec{r}-\vec{d})(\vec{r}-\vec{d}) - (\vec{r}-\vec{d})^2 \overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - 3\vec{r}\,\vec{d} - 3\vec{d}\,\vec{r} + 3\,\vec{d}\,\vec{d} - (r^2 + d^2 - 2\vec{r}\cdot\vec{d})\overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - r^2 \overleftrightarrow{I} \right] - 3\left[\int d^3r\, \rho\, \vec{r} \right] \vec{d} - 3\vec{d} \left[\int d^3r\, \rho\, \vec{r} \right]$$

$$+ 3\vec{d}\,\vec{d} \left[\int d^3r\, \rho \right] - d^2 \overleftrightarrow{I} \left[\int d^3r\, \rho \right]$$

$$+ 2 \left[\int d^3r\, \rho\, \vec{r} \right] \cdot \vec{d}\, \overleftrightarrow{I}$$

$$\tilde{\overleftrightarrow{Q}} = \overleftrightarrow{Q} - 3\vec{p}\,\vec{d} - 3\vec{d}\,\vec{p} + 3\vec{d}\,\vec{d}\, q - \left(d^2 q - 2\vec{p}\cdot\vec{d} \right) \overleftrightarrow{I}$$

we see that $\overleftrightarrow{Q}$ is independent of choice of origin only when both q and $\vec{p}$ vanish. When this happens the quadrupole term is the leading term in the multipole expansion.

In general, the leading term in multipole expansion will be indep of origin of coordinates.