Supposed for some distribution ρ we have the monopole moment $q=0$. $\Rightarrow$ dipole moment $\vec{p}$ is independent of the choice of the coordinate system.

<u>Can we then choose coordinates such that $\overleftrightarrow{Q} = 0$?</u>

$$Q_{ij} = \int d^3r \, \rho(\vec{r}) \left(3 r_i r_j - r^2 \delta_{ij}\right)$$

$\overleftrightarrow{Q}$ is not only symmetric, ie $Q_{ij} = Q_{ji}$, but it is traceless $\sum_i Q_{ii} = Q_{xx} + Q_{yy} + Q_{zz} = 0$

<u>proof</u>: $\sum_i Q_{ii} = \int d^3r \, \rho(\vec{r}) \left[3 \sum_i r_i r_i - r^2 \sum_i \delta_{ii} \right]$

$$= \int d^3r \, \rho(\vec{r}) \left[3r^2 - r^2 (3) \right] = 0$$

So there are really only 5 independent components to $\overleftrightarrow{Q}$.

<u>But</u> since $\overleftrightarrow{Q}$ is symmetric, we know that we can always diagonalize the matrix Q_{ij} and its eigenvalues are real. Or equivalently, we can always rotate our orthonormal coordinate system so that $\overleftrightarrow{Q}$ is diagonal in that coordinate system

$$\begin{pmatrix} Q_{xx} & 0 & 0 \\ 0 & Q_{yy} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}$$

and if $\overleftrightarrow{Q}$ is traceless in one coord system, it is traceless in all coordinate systems $\Rightarrow Q_{xx} + Q_{yy} + Q_{zz} = 0$

$\Rightarrow$ only <u>two</u> independent components in the diagonalizing frame

Since we have three degrees of freedom d_x, d_y, d_z in translating to a new origin, one might think that we can always choose a ~~new~~ new coordinate system in which $Q_{xx} = Q_{yy} = 0$ and then by traceless condition $Q_{zz} = 0$ also and so $\overleftrightarrow{Q} = 0$ (if all eigenvalues are zero, the matrix must vanish)

Under a shift of coordinates $\tilde{\vec{r}} = \vec{r} - \vec{d}$, the new quadrupole tensor is related to the old by

$$\tilde{\overleftrightarrow{Q}} = \overleftrightarrow{Q} - 3\vec{p}\vec{d} - 3\vec{d}\vec{p} + 3\vec{d}\vec{d}q - (d^2 q - 2\vec{p}\cdot\vec{d})\,\underline{\underline{I}}$$

if, as assumed, $q = 0$, then

$$\tilde{\overleftrightarrow{Q}} = \overleftrightarrow{Q} - 3\vec{p}\vec{d} - 3\vec{d}\vec{p} + 2\vec{p}\cdot\vec{d}\,\overleftrightarrow{I}$$

Suppose we start in a frame in which $\overleftrightarrow{Q}$ is diagonal, ie only $Q_{xx}, Q_{yy}, Q_{zz} \neq 0$ then we have the transformations

1) $\tilde{Q}_{xx} = Q_{xx} - 4p_x d_x + 2p_y d_y + 2p_z d_z$
2) $\tilde{Q}_{yy} = Q_{yy} + 2p_x d_x - 4p_y d_y + 2p_z d_z$
3) $\tilde{Q}_{zz} = Q_{zz} + 2p_x d_x + 2p_y d_y - 4p_z d_z$

4) $\tilde{Q}_{xy} = -3(p_x d_y + p_y d_x)$
5) $\tilde{Q}_{yz} = -3(p_y d_z + p_z d_y)$
6) $\tilde{Q}_{zx} = -3(p_z d_x + p_x d_z)$

we have 6 linear equations in three unknowns d_x, d_y, d_z

Since we know $\overleftarrow{Q}$ is always traceless then we can eliminate one of equations (1), (2), and (3) since $Q_{zz} = -(Q_{xx} + Q_{yy})$ so (3) is dependent on (1) and (2)

That gives 5 equations.

Can we choose d_x, d_y, d_z so that $\tilde{Q}_{xx} = \tilde{Q}_{yy} = \tilde{Q}_{xy} = \tilde{Q}_{yz} = \tilde{Q}_{zx} = 0$?

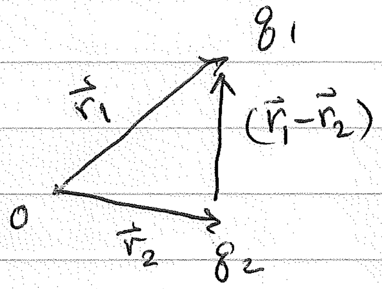
In general NO! that would be 5 equations in 3 unknowns so the system is in general overspecified and there is no solution. Only in special cases might there be a solution.

Note, even though $Q_{xy} = Q_{yz} = Q_{zx} = 0$ in the original coordinate system that does not general remain so in the translated coordinate system

In general we cannot use rotations to rotate a non-zero tensor into a zero tensor. This is why adding the rotational degrees of freedom to our coordinate transformation does not help.

Example two charges q_1 at $\vec{r}_1$ and q_2 at $\vec{r}_2$

$$q_1 + q_2 = q \neq 0$$



monopole $q_1 + q_2 = q$

dipole $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $\overleftrightarrow{Q} = (3\vec{r}_1\vec{r}_1 - r_1^2 \overleftrightarrow{I}) q_1 + (3\vec{r}_2\vec{r}_2 - r_2^2 \overleftrightarrow{I}) q_2$

We can make the dipole moment vanish by shifting to a new coord system $\vec{r}\,' = \vec{r} - \vec{d}$ where $\vec{d} = \dfrac{\vec{p}}{q}$

$$\vec{r}\,' = \vec{r} - \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2} = \frac{q_1(\vec{r} - \vec{r}_1) + q_2(\vec{r} - \vec{r}_2)}{q_1 + q_2}$$

positions of q_1, q_2 in new coords are

$$\vec{r}_1\,' = \frac{q_2}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

lies along vector from $\vec{r}_2$ to $\vec{r}_1$

Origin of new coord system is at

$$\vec{r}\,' = 0 \Rightarrow \vec{r} = \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2}$$ "center of charge"

for many charges q_i at positions $\vec{r}_i$, the origin that makes dipole moment vanish is at

$$\vec{r} = \frac{\sum_i q_i \vec{r}_i}{\sum_i q_i}$$

In this coord system

$$\vec{p}\,' = q_1 \vec{r}_1\,' + q_2 \vec{r}_2\,' = \frac{q_1 q_2}{q_1+q_2}(\vec{r}_1 - \vec{r}_2) - \frac{q_2 q_1}{q_1+q_2}(\vec{r}_1 - \vec{r}_2)$$

$$= 0 \quad \text{as it must be!}$$

<u>Quadrupole moment</u> in the coord system in which $\vec{p}\,' = 0$ the quadrupole tensor is

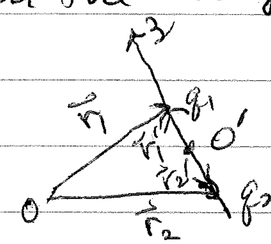
$$\overleftrightarrow{Q}\,' = \left[3\vec{r}_1\,'\vec{r}_1\,' - (r_1')^2 \overleftrightarrow{I}\right] q_1 + \left[3\vec{r}_2\,'\vec{r}_2\,' - (r_2')^2 \overleftrightarrow{I}\right] q_2$$

let us choose ~~coord~~ spherical coordinates with origin at O' and $\hat{z}$ axis aligned along $\vec{r}_1 - \vec{r}_2$, so that

$$\vec{r}_1 - \vec{r}_2 = s\,\hat{z} \quad \text{where } s = |\vec{r}_1 - \vec{r}_2| \text{ is separation between the charges}$$

Then $$\vec{r}_1\,' = \frac{q_2}{q_1+q_2}\, s\,\hat{z}$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1+q_2}\, s\,\hat{z}$$



$$\overleftrightarrow{Q}\,' = \left(\frac{q_2}{q_1+q_2}\right)^2 q_1 \left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$

$$+ \left(\frac{-q_1}{q_1+q_2}\right)^2 q_2 \left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$

$$\overleftrightarrow{Q}' = \frac{q_2^2 q_1 + q_1^2 q_2}{(q_1+q_2)^2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I}\right]$$

$$= \frac{q_1 q_2}{q_1+q_2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I}\right]$$

$$Q'_{ij} = \frac{q_1 q_2}{q_1+q_2} s^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ in xyz coord system

as $\hat{z}\hat{z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(check: $\overleftrightarrow{Q}'$ is traceless $-1-1+2=0$)

The contribution of quadrupole to the potential is

$$\Phi_{quad} = \frac{1}{2} \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{r^3} \qquad \hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

with origin at O' this becomes — in xyz coords

$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

do matrix multiplications

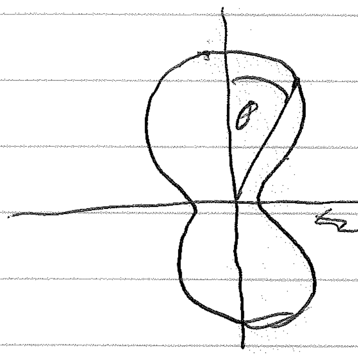
$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (2\cos^2\theta - \sin^2\theta)$$

independent of φ as it must be due to azimuthal symmetry

$$\Phi_{quad} = \frac{s^2}{2r^3}\,\frac{q_1 q_2}{q_1+q_2}\left(2\cos^2\theta - \sin^2\theta\right)$$

use $\sin^2\theta = 1-\cos^2\theta \Rightarrow 2\cos^2\theta - \sin^2\theta = 3\cos^2\theta - 1$

use $\cos^2\theta = \dfrac{1+\cos 2\theta}{2} \Rightarrow 3\cos^2\theta - 1 = \dfrac{1+3\cos 2\theta}{2}$

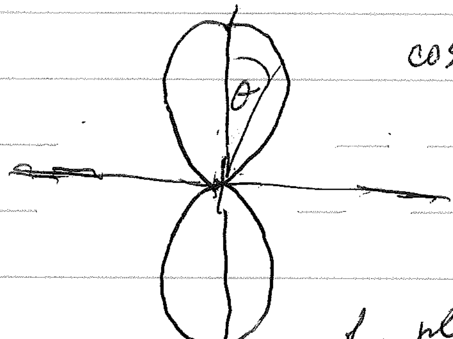


$\dfrac{1+3\cos 2\theta}{2}$

does not go to zero at $\theta = \pi/2$

polar plot

Compare to dipole $\Phi_{dip} = \dfrac{p\cos\theta}{r^2}$



$\cos\theta$

polar plot

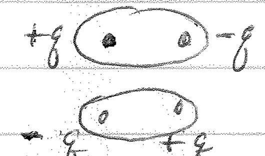
Example

sample charge config's

o q $\Rightarrow$ monopole is leading term

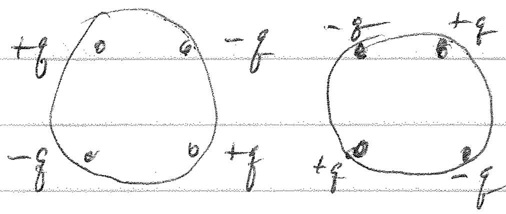
$+q$ o o $-q$ $\Rightarrow$ monopole $=0$ $\Rightarrow$ dipole is leading term

$\vec{p}$ is indep of origin



$\Rightarrow$ monopole $=0$ $\Rightarrow$ total dipole is sum of dipoles of individual neutral pairs

$=0$ leading term is quadrapole



when monopole $=0$ and dipole $=0$, quadrapole is indep of origin.
$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q=0$ and $\vec{p}=0$

$Q = Q_1 + Q_2$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is octopole

Magnetostatics

$$\vec{B} = \vec{\nabla}\times\vec{A}\ , \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{j} \end{cases}$$ Ampere's law (statics only!)

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a <u>simple</u> expression in <u>Cartesian coords</u>

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2\vec{A} = \nabla^2(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi})$$
$$\cancel{= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta}) + (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}} \quad \text{Poisson's equation!}$$

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \; \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \qquad \text{three equations for } A_x,\ A_y,\ A_z \ !$$

for localized current sources $j(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\left(r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2\right)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1^{st} order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}\,'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \overset{①}{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}} + \overset{②}{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r}\cdot\vec{r}')}{r^3}} + \cdots$$

Consider term ①

$$\int d^3r \, \vec{j}(\vec{r}) \qquad \int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\int d^3r \, j_i(r) = \sum_{j=1}^{3} \int d^3r \, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

for i^{th} component

$$= \sum_j \left\{ \oint_S da \, j_j \, r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r \, \vec{j}(\vec{r}) = 0$ in magnetostatics

monopole term vanishes

term (2)

$$\int d^3r \; \vec{j}(\vec{r})\,\vec{r} \qquad \text{tensor} \qquad \frac{\partial r_i}{\partial r_k} = \delta_{ik}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

ij^{th} element of tensor

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{j} = 0$ in magnetostatics $\qquad \frac{\partial r_j}{\partial r_k} = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \, j_i r_j = -\int d^3r \, j_j r_i$

$$= \frac{1}{2}\int d^3r \left(j_i r_j - j_j r_i \right)$$

Going back to term (2) in expansion for $\vec{A}$

So

$$\int d^3r' \, j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_{j=1}^{3} r_j \int d^3r' \, j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2} \int d^3r' \left(j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2} \int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{j}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{J}) = \vec{r}'(\vec{r} \cdot \vec{J}) - \vec{J}(\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \vec{J}(\hat{r} \cdot \vec{r}') = -\frac{1}{2}\hat{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')\right]$$

<u>define</u> the magnetic dipole moment as

$$\boxed{\vec{m} \equiv \frac{1}{2c}\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')}$$

then in the magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}_{dip}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

what is the magnetic field in this approx?

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3}\right)$$

to do the double cross product, it is convenient to use the Levi-Civita symbol ε_{ijk} defined as

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{any two of the indices are equal} \\ +1 & ijk \text{ are an } \underline{\text{even}} \text{ permutation of } 123 \\ -1 & ijk \text{ are an } \underline{\text{odd}} \text{ permutation of } 123 \end{cases}$$

$i,j,k = 1,2,3$

In terms of Levi-Civita symbol $(\vec{A} \times \vec{B})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$

check by writing out $\vec{A} \times \vec{B}$ in terms of components

Summation convension: when ever we have a pair of indices repeated, we sum over them, so

$$\varepsilon_{ijk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$$

index j appears twice
index k appears twice

A ~~very~~ useful identity:

Kronecker deltas

$$\boxed{\varepsilon_{kij}\, \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\,\delta_{jl}}$$

So we now put this to use!

$$\vec{B}_{dip} = \vec{\nabla}\times\left(\vec{m}\times\frac{\vec{r}}{r^3}\right)$$

i component

$$(B_{dip})_i = \varepsilon_{ijk}\,\partial_j\, \varepsilon_{klm}\, m_l \frac{r_m}{r^3} \qquad \text{where } \partial_j \equiv \frac{\partial}{\partial r_j}$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j\left(\frac{m_l r_m}{r^3}\right)$$

$\varepsilon_{ijk} = \varepsilon_{kij}$ as an even permutation takes one to the other

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\left(m_l \frac{r_m}{r^3}\right)$$

$$= m_i\,\partial_j\left(\frac{r_j}{r^3}\right) - m_j\,\partial_j\left(\frac{r_i}{r^3}\right)$$

Now $\partial_j\left(\frac{r_j}{r^3}\right) = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r})$

$$\partial_j\left(\frac{r_i}{r^3}\right) = \frac{1}{r^3}\frac{\partial r_i}{\partial r_j} - \frac{3r_i}{r^4}\frac{\partial r}{\partial r_j}$$

by product rule and $\frac{\partial r_i}{\partial r_j} = \delta_{ij}$

Now $\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}$ so $\frac{\partial r}{\partial r_j} = \frac{r_j}{r}$

don't care about this term since we only want $\vec{B}$ far away from current where $\delta(\vec{r}) = 0$.

so

$$(B_{dip})_i = m_i 4\pi \delta(\vec{r}) - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i r_j}{r^5} \right]$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\, r_i}{r^5}$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\hat{r})\, \hat{r}_i}{r^3}$$

so

$$\boxed{\vec{B}_{dip} = \frac{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}}{r^3}}$$