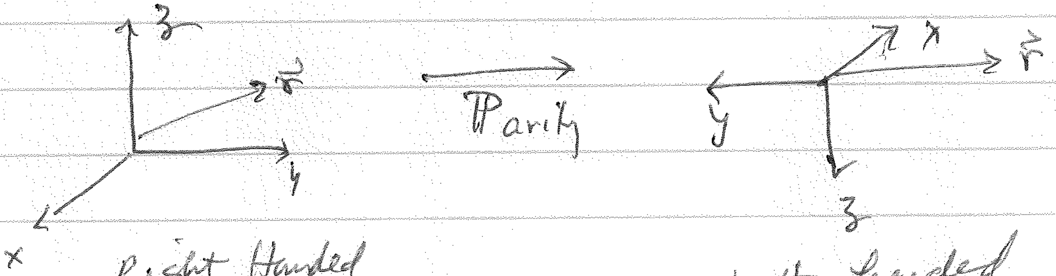
Symmetry under parity transformation

vector vs. pseudo vector



Right Handed   Left handed

$\vec r = (x, y, z) \longrightarrow (-x, -y, -z)$

$\mathbb{P}(\vec r) = -\vec r$ position $\vec r$ is odd under parity

Any vector-like quantity that is odd under $\mathbb{P}$ is a vector.

examples of vectors

position $\vec r$

velocity $\vec v = \frac{d\vec r}{dt}$ since $\vec r$ is vector and t is scalar $\mathbb{P}(t) = t$

acceleration $\vec a = \frac{d\vec v}{dt}$

Force $\vec F = m\vec a$ since $\vec a$ is vector and m is scalar

momentum $\vec p = m\vec v$.   since $\vec v$ is vector and m is scalar

electric field $\vec F = q\vec E$ since $\vec F$ is vector and q is scalar $\mathbb{P}(q) = q$

current $\vec J = \sum_i q_i \vec v_i \, \delta(\vec r - \vec r_i(t))$

any vector-like quantity that is even under $\mathbb{P}$ is a psuedovector

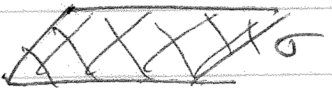
angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \to -\vec{r}$ and $\vec{p} \to -\vec{p}$,
$\vec{L} \to \vec{L}$ under $\mathbb{P}$
$\vec{L}$ is even under $\mathbb{P}$

magnetic field $\vec{F} = q\vec{v} \times \vec{B}$ since $\vec{F}$ and $\vec{v}$ are vectors and q is scalar, $\vec{B}$ must be ~~psuedo~~ psuedovector.

cross product of any two vectors is a psuedovector
" " " vector and psuedovector is a vector

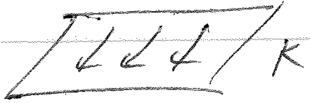
when solving for $\vec{E}$, it can only ~~be~~ ~~E~~ made up of vectors that exist in the problem
When solving for $\vec{B}$, it can only be made up of psuedovectors that exist in the problem

ex charged plane



only directions in problem is normal $\hat{n}$
$\hat{n}$ is a vector
$\vec{E} \propto \hat{n}$

surface current



only directions are the vectors $\hat{n}$ and $\vec{K}$. But $\vec{B}$ can only be made of psuedovectors

$$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{n})$$

Dielectrics + Magnetic Materials - Macroscopic Maxwell Equ

Dielectrics

Maxwell's equations apply exactly to the true microscopic electric ad magnetic fields that arise from all charges ad currents.

$$\vec{\nabla}\cdot\vec{b} = 0 \qquad \vec{\nabla}\times\vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho_0 \qquad \vec{\nabla}\times\vec{b} = \frac{4\pi}{c}\vec{j}_0 + \frac{1}{c}\frac{\partial\vec{e}}{\partial t}$$

where $\vec{e}$ ad $\vec{b}$ are microscopic fields from total charge density ρ_0 ad current density $\vec{j}_0$.

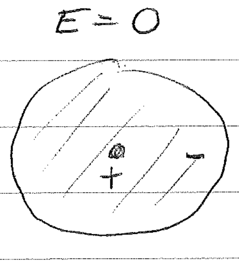
However, in most problems involving macroscopic objects, if we took ρ_0 ad $\vec{j}_0$ to describe charge + current of each individual atom in a material, then they, and the resulting $\vec{e}$ ad $\vec{b}$ would be enormously complicated functions varying rapidly over distances $\sim 10^{-8}$ cm ad times $\sim 10^{-16}$ sec.

In classical E+M we are generally concerned with phenomena that vary extremely slowly compared to these length + time scales, Rather than worry about the microscopic details of ρ and $\vec{j}$ and resulting $\vec{e}$ and $\vec{b}$ we want to describe phenomena in terms of ~~averaged~~ ~~smoothly varying quan~~ averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the macroscopic Maxwell equations.

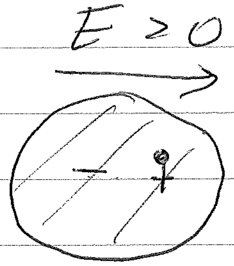
Dielectric Materials

can be solid, liquid or gas

A dielectric material is an insulator. valence Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged ρ in the dielectric vanishes! One might therefore think that electrodynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However this is not true due to the phenomena of "polarization".



electron cloud
centered on ionic
~~nucle~~ core
dipole moment
vanishes



electron cloud
and ionic core
displaced $\vec{d} \propto \vec{E}$
atom is "polarized"
has dipole moment $\vec{p} = q\vec{d} \propto q\vec{E}$

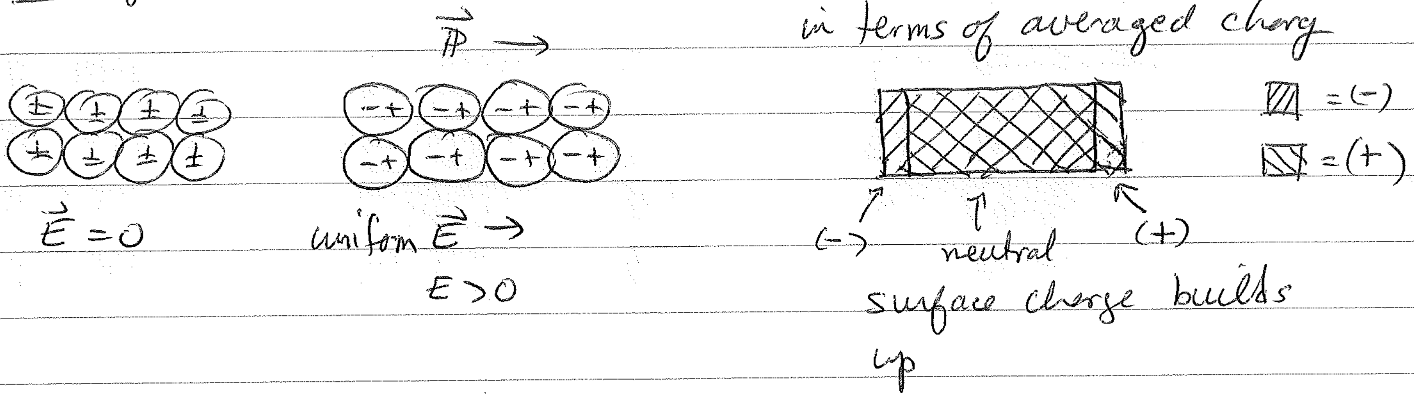
$\vec{p} = \underset{\uparrow}{\alpha} \vec{E}$
atomic polarizability

Polarization density $\vec{\mathbb{P}}(\vec{r}) = \sum_i \underset{\uparrow}{\vec{p}_i}\, \delta(\vec{r}-\vec{r}_i)$

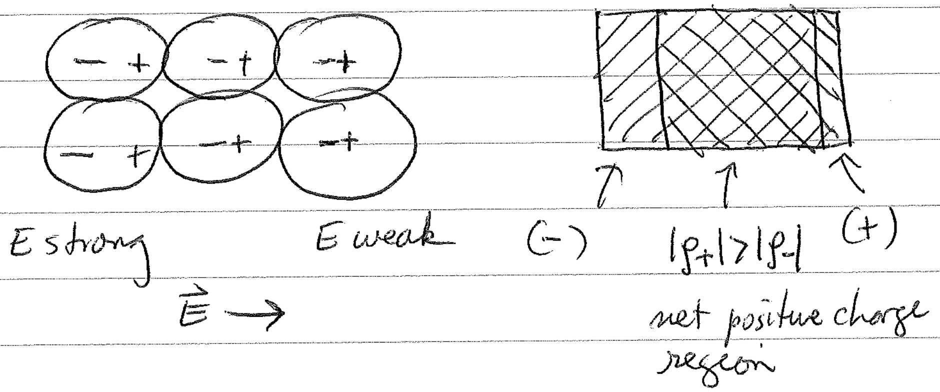
dipole moment of atom i
at position $\vec{r}_i$

Polarization density $\vec{\mathbb{P}}$ can give rise to regions of net charge – sometimes called "bound charge"

<u>Example</u>



For a non uniform $\vec{E}$, atoms are more strongly polarized where E is largest



For uniform $\vec{\mathbb{P}}$, build up surface charge σ_b
For nonuniform $\vec{\mathbb{P}}$, also can build up vol charge density ρ_b.

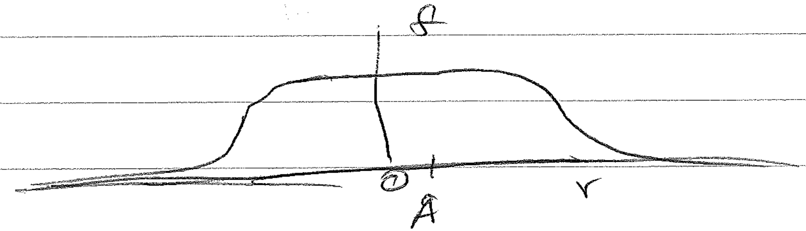
We now carry out the averaging explicitly to see how such polarization enters the macroscopic Maxwell equations

(Jackson 6.6)

Define spatially averaged quantities by

$$\langle F(\vec{x},t)\rangle = \int d^3r'\, f(\vec{r}')\, F(\vec{r}-\vec{r}',t)$$

where $f(\vec{r})$ vanishes for $|\vec{r}|$ large on microscopic length scales, but short on macroscopic length scales. $f(\vec{r})$ normalized to unity $\int d^3r\, f(\vec{r}) = 1$.
Other details of $f(\vec{r})$ are not too important, as long as $f(\vec{r})$ is a smooth function of $\vec{r}$



want $f \approx 1$ for $r < R$
$f \approx 0$ for $r \gg R$
where R is length scale inbetween micro + macro

$$\frac{\partial}{\partial r_i}\langle F(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\,\frac{\partial F(r-r')}{\partial r_i} = \left\langle \frac{\partial F}{\partial r_i}\right\rangle$$

$$\frac{\partial}{\partial t}\langle F(\vec{r},t)\rangle = \left\langle \frac{\partial F}{\partial t}\right\rangle$$

Define the macroscopic fields

$$\vec{E}(r,t) \equiv \langle \vec{e}(\vec{r},t)\rangle$$
$$\vec{B}(\vec{r},t) \equiv \langle \vec{b}(\vec{r},t)\rangle$$

Then $\vec{\nabla}\cdot\vec{b} = 0 \Rightarrow \langle \vec{\nabla}\cdot\vec{b} \rangle = 0$

$$\Rightarrow \vec{\nabla}\cdot\langle\vec{b}\rangle = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{e} + \frac{\partial \vec{b}}{c\partial t} = 0 \Rightarrow \vec{\nabla}\times\langle\vec{e}\rangle + \frac{\partial}{c\partial t}\langle\vec{b}\rangle = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{c\partial t} = 0$$

Remaining Maxwell eqn, upon averaging, become

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle\rho_0\rangle$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle\vec{j}_0\rangle + \frac{\partial\vec{E}}{c\partial t}$$

Consider $\langle\rho_0\rangle$

$\rho_0 = \sum_i q_i\,\delta(\vec{r}-\vec{r}_i(t))$ sum over all charges

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric

$\rho_{free} = \sum_{i\ free} q_i\,\delta(\vec{r}-\vec{r}_i(t))$ sum over only free charges ie, external charge added to the material

$\rho_{bound} = \sum_n \rho_n(\vec{r},t)$

↖ charge distribution of molecule n

$\rho_n(\vec{r},t) = \sum_{i\in n} q_i\,\delta(\vec{r}-\vec{r}_i(t))$ sum over charges in molecule n

$$\langle \rho_n(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\, \rho_n(\vec{r}-\vec{r}',t)$$

$$= \sum_{i\in n} q_i \int d^3r'\, f(\vec{r}')\, \delta(\vec{r}-\vec{r}'-\vec{r}_i(t)\,)\,)$$

$$= \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_i(t))$$

write $\vec{r}_i(t) = \vec{r}_n(t) + \vec{r}_{ni}(t)$

$\uparrow$ position of center of mass of molecule n

$\uparrow$ position of charge i of molecule n with respect to center of mass

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_n-\vec{r}_{ni})$$

Since the $|\vec{r}_{ni}|$ are all of atomic length scale, and f is slowly varying on this length scale, we can expand

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i \Big[f(\vec{r}-\vec{r}_n) - \big(\vec{\nabla} f(\vec{r}-\vec{r}_n)\big)\cdot \vec{r}_{ni} + \frac{1}{2}\sum_{\alpha\beta=1}^{3} \frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha\, \partial r_\beta} (\vec{r}_{ni})_\alpha (\vec{r}_{ni})_\beta + \cdots \Big]$$

$\alpha, \beta = 1,2,3$ or x,y,z

$$= f(\vec{r}-\vec{r}_n)\Big[\sum_{i\in n} q_i\Big]$$

$$- \big(\vec{\nabla} f(\vec{r}-\vec{r}_n)\big)\cdot \sum_{i\in n} q_i \vec{r}_{ni}$$

$$+ \sum_{\alpha,\beta=1}^{3} \left(\frac{1}{6}\frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha\, \partial r_\beta}\right) 3\sum_{i\in n} q_i (r_{ni})_\alpha (r_{ni})_\beta$$

Define $q_n \equiv \sum_{i\in n} q_i$ total charge molecule n

$\vec{p}_n \equiv \sum_{i\in n} q_i \vec{r}_{ni}$ dipole moment about center of mass of molec n

$\overleftrightarrow{Q}_n' \equiv \sum_{i\in n} 3 q_i \vec{r}_{ni} \vec{r}_{ni}$ quadrapole moment about center of mass of molec n

(prime on $\overleftrightarrow{Q}'$ since definition here is a little different from that of multipole exp)

$$\langle \rho_n(\vec{r},t)\rangle = q_n f(\vec{r}-\vec{r}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)$$

$$+ \frac{1}{6} \sum_{\alpha\beta} (Q_n')_{\alpha\beta} \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$

Now use $\langle \delta(\vec{r}-\vec{r}_n)\rangle = f(\vec{r}-\vec{r}_n)$ by definition of averaging

$$\Rightarrow \langle \rho_n(\vec{r},t)\rangle = \langle q_n \delta(\vec{r}-\vec{r}_n)\rangle$$

$$- \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle \qquad \vec{\nabla}\cdot p\,\delta = \vec{p}\cdot\vec{\nabla}\delta$$

$$+ \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \langle (Q_n')_{\alpha\beta}\, \delta(\vec{r}-\vec{r}_n)\rangle$$

Now

$$\langle \rho_{\text{bound}}(\vec{r},t)\rangle = \sum_n \langle \rho_n(\vec{r},t)\rangle$$

$$= \langle \sum_n q_n \delta(\vec{r}-\vec{r}_n)\rangle - \vec{\nabla}\cdot\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$$

$$+ \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \langle \sum_n (Q_n)'_{\alpha\beta}\, \delta(\vec{r}-\vec{r}_n)\rangle$$

Define $\vec{\mathbb{P}}(\vec{r},t) \equiv \langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle$ average polarization density

$\overleftrightarrow{\mathbb{Q}}'(\vec{r},t) \equiv \frac{1}{6}\langle \sum_n \overleftrightarrow{Q}'_n \delta(\vec{r}-\vec{r}_n) \rangle$ average quadrapole density

$$\langle \rho_{bound} \rangle = \sum_n \langle q_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{\nabla}\cdot\vec{\mathbb{P}} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftrightarrow{\mathbb{Q}}'_{\alpha\beta}$$

Define the macroscopic charge density

$$\rho \equiv \langle \sum_{i\ free} q_i \delta(\vec{r}-\vec{r}_i) \rangle + \langle \sum_{\substack{n \\ molec}} q_n \delta(\vec{r}-\vec{r}_n) \rangle$$

↖ vanishes if molecule is charge neutral

Then

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle \rho_0 \rangle = 4\pi\left[\rho - \vec{\nabla}\cdot\vec{\mathbb{P}} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftrightarrow{\mathbb{Q}}'_{\alpha\beta}\right]$$

$$\sum_\alpha \frac{\partial}{\partial r_\alpha}\left[E_\alpha + 4\pi \mathbb{P}_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} \mathbb{Q}'_{\alpha\beta}\right] = 4\pi\rho$$

Define electric displacement vector

$$D_\alpha = E_\alpha + 4\pi \mathbb{P}_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} \mathbb{Q}'_{\alpha\beta}$$

then $\boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho}$

In most materials, the quadrapole and higher terms are negligible and we can take

$$\vec{D} = \vec{E} + 4\pi\vec{\mathbb{P}}$$