why quadrupole + higher order terms can generally be ignored:

Let a_0 be the length scale that characterizes the size of a molecule in the dielectric
Let ℓ be the typical spacing between molecules
Let L be the length scale of the spatial averaging function $f(\vec{r})$. $L \gg a_0$

dipole moment per a_0

$\Rightarrow$ polarization density $\mathbb{P} \sim \frac{a_0}{\ell^3}$

$$\vec{\nabla} \cdot \vec{\mathbb{P}} \sim \left(\frac{a_0}{L}\right) \frac{1}{\ell^3}$$

since $\mathbb{P}$ cannot vary on length scale shorter than the averaging length L

quadrupole moment

$$Q \sim a_0^2$$

quadrupole density $\mathbb{Q} \sim \frac{a_0^2}{\ell^3}$

$$\frac{\partial^2 \mathbb{Q}}{\partial r_\alpha \partial r_\beta} \sim \left(\frac{a_0}{L}\right)^2 \frac{1}{\ell^3}$$

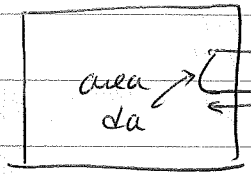
each higher moment gives extra factor a_0
each higher derivative gives extra factor $\frac{1}{L}$

So quadrupole is smaller than dipole term by factor $\left(\frac{a_0}{L}\right) \ll 1$. Higher terms smaller by additional factors of $\left(\frac{a_0}{L}\right)$

For insulators one generally has $q_n = 0$
molecules are neutral.
In this case the macroscopic ρ is just the free charge $\rho = \langle \rho_{free} \rangle$.
And the bound charge is just

$$\boxed{\langle \rho_{bound} \rangle = -\vec{\nabla} \cdot \vec{P}}$$

At a surface of a dielectric



Gaussian pillbox V

$$-\int_V d^3r \; \vec{\nabla} \cdot \vec{P} = -\oint_S da \; \hat{n} \cdot \vec{P}$$

contrib from sides $\to 0$ as $\ell \to 0$
contrib from outside surface $= 0$ as $P = 0$ outside

$$= \hat{n} \cdot \vec{P} \, da$$

only contrib is from inside surface
($\hat{n}$ is outward normal

$$= \int_V d^3r \; \rho_{bound}$$

as $\ell \to 0$, $\int_V d^3r \; \rho_{bound} \to \int da \; \sigma_{bound} = da \, \sigma_{bound}$ surface charge

$$\Rightarrow \boxed{\sigma_{bound} = \hat{n} \cdot \vec{P}}$$ at surface of dielectric

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The

Magnetic Materials

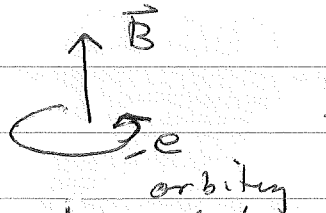
Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:

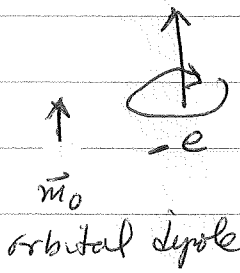
1) intrinsic angular momentum of electrons, ie "electron spin" - can add up and give a net angular momentum to atom
2) orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) ⇒ atoms can have a net magnetic dipole moment. When $\vec{B}=0$, these atomic moments are generally in random orientations (and average to zero, exception is a ferromagnet where moments can allign even if $\vec{B}=0$)
When apply $\vec{B}\neq 0$, the moments tend to allign parallel to $\vec{B}$ giving a net magnetization density $\vec{M} \propto \vec{B}$. This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, ie exists for atoms with zero net dipole moment)



← applying $\vec{B}$ to orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment $\Delta\vec{m} \propto -\vec{B}$



← applying $\vec{B}$ to orbiting electron slows down its orbital velocity. Net result is again that $\Delta\vec{m} \propto -\vec{B}$

see Griffiths chpt 6 + prob 7.17 2nd ed for details

No matter which way electron orbits with respect to $\vec{B}$, result is a decrease in magnetic moment, so $\Delta\vec{m} \propto -\vec{B}$

That $\Delta\vec{m}$ is opposite to $\vec{B}$ is called Lenz magnetism

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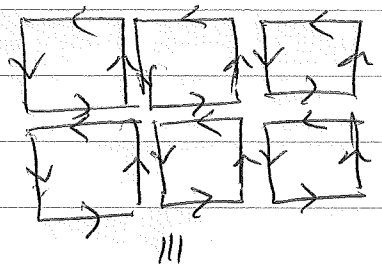
Model atomic magnetic moments as small current loops. When loops get oriented, ie there is non zero average magnetization density

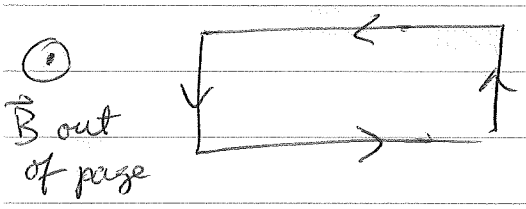
$$\vec{M}(\vec{r}) = \sum_i \vec{m}_i \, \delta(\vec{r} - \vec{r}_i)$$

then net effect is to have a current flowing around the system. This current gives rise to magnetic fields

aligned atomic moments in a uniform applied $\vec{B}$

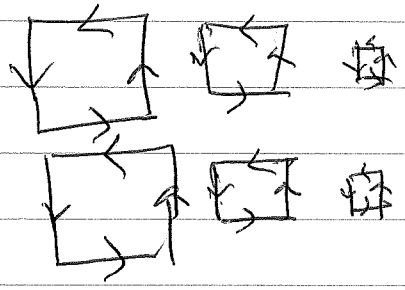


in interior, currents in opposite directions cancel also $\vec{j} = 0$ inside



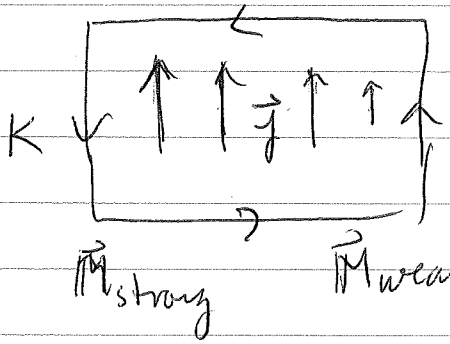
but is net circulation of current around boundary of material
$\Rightarrow$ surface current $\vec{K}_{bound}$

If $\vec{B}$ is not uniform, then $\vec{M}$ is not uniform
Can create finite current density $\vec{j}$ in interior, as well as surface currents



Now currents in interior do not cancell. Net current $\vec{j}_{bound}$ in interior

$\vec{B}$ strong $\vec{B}$ weak



$\vec{B}$ out of page $\Rightarrow$ $\vec{M}$ out of page
~~$\vec{M}$ varies along page~~
$\vec{M}$ varies in direction $\perp$ direction of $\vec{M}$
$\Rightarrow \vec{\nabla} \times \vec{M} \neq 0$ gives $\vec{j}_{bound}$

Average current

$$\langle \vec{j}_0 \rangle = \Big\langle \sum_{i \in \text{free}} q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i) \Big\rangle + \sum_{n}^{\text{molec}} \langle \vec{j}_n \rangle$$

- $\sum_{i \in \text{free}} q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i)$: current from free charges
- $\sum_{n}^{\text{molec}} \langle \vec{j}_n \rangle$: current from molecule n of the dielectric

$$\langle \vec{j}_n(\vec{r}, t) \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \langle \delta(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t)) \rangle$$

$$= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) f(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t))$$

- $\vec{v}_n = \frac{d\vec{r}_n}{dt}$
- $\vec{v}_{ni} = \frac{d\vec{r}_{ni}}{dt}$
- $\vec{r}_n$: position of CM of molec n
- $\vec{r}_{ni}$: position of charge i wrt CM

as with $\langle \rho_0 \rangle$, we can expand in $\vec{r}_{ni}$

$$\langle \vec{j}_n \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \Big\{ f(\vec{r} - \vec{r}_n) - \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n)$$

$$+ \frac{1}{2} \sum_{\alpha\beta} (r_{ni})_\alpha (r_{ni})_\beta \frac{\partial^2 f(\vec{r} - \vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$

$$+ \cdots \Big\}$$

we will keep only the first two terms in the expansion

The various terms we have to consider are

① $\sum_{i \in n} q_i \vec{v}_n f(\vec{r}-\vec{r}_n)$

② $\sum_{i \in n} q_i \vec{v}_{ni} \delta(\vec{r}-\vec{r}_n)$

③ $-\sum_{i \in n} q_i \vec{v}_n \left[\vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

④ $-\sum_{i \in n} q_i \vec{v}_{ni} \left[\vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

① $= \vec{v}_n f(\vec{r}-\vec{r}_n) \sum_{i \in n} q_i = q_n \vec{v}_n f(r-r_n)$

$= \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle$

this is just current of molecule as if it were a point charge q_n. For a neutral molecule $q_n = 0$ and this term vanishes.

② Note: $\frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle = \frac{\partial}{\partial t} \left(\sum_{i \in n} q_i \vec{r}_{ni} f(\vec{r}-\vec{r}_n) \right)$

$= \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

$+ \sum_{i \in n} q_i \vec{r}_{ni} \left[\underbrace{-\vec{\nabla} f(\vec{r}-\vec{r}_n) \cdot \vec{v}_n}_{= \frac{\partial f(\vec{r}-\vec{r}_n)}{\partial t}} \right]$

So for ②, $\sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

$= \frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle$

$+ \left[\vec{v}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right] \vec{p}_n$

So

$$② = \sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n) = \frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle + (\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(r-r_n)\rangle$$

$$2^{nd}\text{ term is } \sum_\alpha v_{n\alpha}\frac{\partial}{\partial r_\alpha}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$$

$$③ = -\vec{v}_n\left(\sum_{i\in n} q_i \vec{r}_{ni}\right)\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n) = -\vec{v}_n\left(\vec{p}_n\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)$$

$$= -\vec{v}_n\,\vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle = \sum_\alpha \vec{v}_n \frac{\partial}{\partial r_\alpha}\langle p_{n\alpha}\delta(\vec{r}-\vec{r}_n)\rangle$$

$$④ = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni}$$

We have seen the tensor $\sum_i q_i \vec{r}_{ni}\vec{v}_{ni}$ before when we considered the magnetic dipole moment

$$\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni} = \int d^3r\, \vec{r}\,\vec{j} \qquad \text{where } \vec{j}(\vec{r}) \equiv \sum_{i\in n} q_i \vec{v}_{ni}\delta(\vec{r}-\vec{r}_{ni})$$

is current density with respect to center of mass of molecule

We had $\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} - \int d^3r\,(\vec{\nabla}\cdot\vec{j})\,\vec{r}\,\vec{r}$

↑ in statics, $\vec{\nabla}\cdot\vec{j} = 0$

in general $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$

$$\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} + \int d^3r \frac{\partial \rho}{\partial t}\,\vec{r}\,\vec{r}$$

$$= -\int d^3r\, \vec{j}\,\vec{r} + \frac{\partial}{\partial t}\left[\int d^3r\, \rho\,\vec{r}\,\vec{r}\right]$$

↑ although this is not zero, it is a quadrapole term of the same order as the terms we dropped when we truncated expansion to linear order

$$\sim O\left(\frac{a_0}{L}\right)^2$$

So $\int d^3r\ \vec{r}\,\vec{f} \simeq -\int d^3r\ \vec{f}\,\vec{r}$ ignoring the quadrupole term

$$= \frac{1}{2}\int d^3r \left[\vec{r}\,\vec{f} - \vec{f}\,\vec{r}\right]$$

$$\sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = \frac{1}{2}\sum_{i\in n} q_i\left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni}\right]$$

$$-\vec{\nabla} f(r-r_n)\cdot\sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot\frac{1}{2}\sum_i q_i\left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni}\right]$$

$$= \frac{-1}{2}\sum_{i\in n} q_i\left[(\vec{\nabla} f\cdot\vec{r}_{ni})\,\vec{v}_{ni} - (\vec{\nabla} f\cdot\vec{v}_{ni})\,\vec{r}_{ni}\right]$$

$$= -\frac{1}{2}\sum_{i\in n} q_i\ \vec{\nabla} f\times(\vec{v}_{ni}\times\vec{r}_{ni}) \qquad \text{triple product rule}$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times\frac{1}{2}\sum_{i\in n}\vec{r}_{ni}\times\vec{v}_{ni}\,q_i$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times\frac{1}{2}\int d^3r\ \vec{r}\times\vec{f}$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times c\vec{m}_n \qquad \text{where } \vec{m}_n = \frac{1}{2c}\sum_{i\in n}\vec{r}_{ni}\times\vec{v}_{ni}\,q_i$$

is magnetic dipole moment of molecule n

$$= \vec{\nabla}\times f(\vec{r}-\vec{r}_n)\,c\vec{m}_n$$

$$= \vec{\nabla}\times\left\langle c\vec{m}_n\,\delta(\vec{r}-\vec{r}_n)\right\rangle$$

Adding all the pieces

$$\langle \vec{j}_n \rangle = \underset{①}{\langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle} + c\vec{\nabla}\times \underset{④}{\langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$+ \frac{\partial}{\partial t}\underset{②}{\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle} + \underset{②}{(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$- \underset{③}{\vec{v}_n \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

Define $\vec{M}(\vec{r}) \equiv \sum_n \langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle$ average magnetization density

$\vec{P}(\vec{r}) \equiv \sum_n \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$ polarization density, as before

$$\sum_n \langle \vec{j}_n \rangle = \sum_n \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle + c\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

$$+ \sum_n \left[(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle - \vec{v}_n \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle \right]$$

see Jackson (6.96) for additional electric quadrupole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities $\vec{v}_n$ are usually small, and randomly oriented, so that they average to zero. (see Jackson (6.100) for case of net translation of dielectric, $\vec{v}_n$ = const all n)

Define macroscopic current density

$$\vec{j}(\vec{r},t) = \left\langle \sum_{i \in \text{free}} q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i) \right\rangle + \left\langle \sum_{n}^{\text{molecs}} q_n \vec{v}_n \, \delta(\vec{r}-\vec{r}_n) \right\rangle$$

↑ current of free charges

↑ current of molecular drifting if molecules are charged

Then $\langle \vec{j}_\mu \rangle = \vec{j} + c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$

Ampere's Law becomes upon averaging

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle \vec{j}_\mu \rangle + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$= \frac{4\pi}{c}\vec{j} + 4\pi\vec{\nabla}\times\vec{M} + \frac{4\pi}{c}\frac{\partial \vec{P}}{\partial t} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{B} - 4\pi\vec{M}\right) = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{P}\right)$$

define $\boxed{\vec{H} \equiv \vec{B} - 4\pi\vec{M}}$ to get

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}}$$

$\boxed{\vec{D} = \vec{E} + 4\pi\vec{P}}$ as before

official nomenclature: $\vec{B}$ is the magnetic induction

$\vec{H}$ is the magnetic field

common usage: both $\vec{H}$ and $\vec{B}$ are called magnetic fields

When atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

When molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{J}_{bound} = \sum_n \langle \vec{J}_n \rangle = c\,\vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of <u>bound charge</u>

$$\vec{\nabla} \cdot \vec{J}_{bound} = c\,\vec{\nabla} \cdot (\vec{\nabla} \times \vec{M}) + \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{P})$$

$$= -\frac{\partial \rho_{bound}}{\partial t} \quad \text{where } \rho_{bound} = -\vec{\nabla} \cdot \vec{P} \text{ is bound charge density}$$

So $\boxed{\vec{\nabla} \cdot \vec{J}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla} \cdot \langle \vec{J}_0 \rangle - \frac{\partial \langle \rho_0 \rangle}{\partial t} = 0, \qquad \text{and } \langle \vec{J}_0 \rangle = \vec{J} + \vec{J}_{bound}$$

($\vec{J}$: free current)

$$\langle \rho_0 \rangle = \rho + \rho_{bound}$$

(ρ: free charge)

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved