

Linear Dielectrics

bound charge is proportional to free charge

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot (\chi_e \vec{E}) = -\vec{\nabla} \cdot \left(\frac{\chi_e}{\epsilon} \vec{D} \right)$$

if χ_e (and hence ϵ) is spatially constant, then

$$\rho_b = -\frac{\chi_e}{\epsilon} \vec{\nabla} \cdot \vec{D} = -\frac{\chi_e}{\epsilon} 4\pi \rho$$

$$\boxed{\rho_b = -\frac{4\pi\chi_e}{1+4\pi\chi_e} \rho}$$

when free charge $\rho = 0$,
then $\rho_b = 0$

$$\rho_{\text{total}} = \rho + \rho_b = \rho \left[1 - \frac{4\pi\chi_e}{1+4\pi\chi_e} \right] = \frac{\rho}{1+4\pi\chi_e} = \boxed{\frac{\rho}{\epsilon} = \rho_{\text{total}}}$$

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.

For linear dielectrics

Statics

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{D} = \epsilon \vec{E} \Rightarrow \vec{\nabla} \cdot (\epsilon \vec{E}) = 4\pi\rho$$

If ϵ is constant in space then $\epsilon \vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho/\epsilon = 4\pi\rho_{\text{tot}} \\ \vec{\nabla} \times \vec{E} &= 0 \end{aligned} \left\{ \begin{array}{l} \text{look just like ordinary} \\ \text{electrostatics but} \\ \text{with } \rho \rightarrow \rho/\epsilon \end{array} \right.$$

Alternatively, could write $\vec{E} = \vec{D}/\epsilon$

$$\Rightarrow \vec{\nabla} \times (\vec{D}/\epsilon) = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{D} = 0 \quad \text{when } \epsilon \text{ constant in space}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= 4\pi\rho \\ \vec{\nabla} \times \vec{D} &= 0 \end{aligned} \left\{ \begin{array}{l} \text{looks just like ordinary} \\ \text{electrostatics, but with } \vec{E} \rightarrow \vec{D} \end{array} \right.$$

Complication arises at interface between dielectrics (or between dielectric and vacuum). At interface, ϵ is not constant $\Rightarrow \vec{\nabla} \times \vec{D} \neq 0$.

What we can do is to solve for $\vec{E}$ or $\vec{D}$ inside each dielectric separately, and then use the boundary conditions

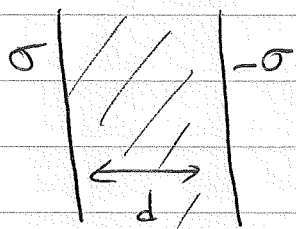
$$\hat{n} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = 4\pi\sigma$$

$$\hat{t} \cdot (\vec{E}^{\text{above}} - \vec{E}^{\text{below}}) = 0$$

to match solutions across the interfaces.

A similar story holds for linear magnetic materials

Single example: parallel plate capacitor filled with a dielectric



σ free charge

What is E between plates?

We know $\vec{E} = \vec{D} = 0$ outside plates

Between plates $\vec{\nabla} \cdot \vec{D} = 0$ as $\rho = 0$

$$\vec{D} = D(x) \hat{x} \Rightarrow \frac{\partial D}{\partial x} = 0 \Rightarrow D \text{ is constant}$$

Boundary conditions:

left side plate

$$\hat{n} = \hat{x} \quad D=0$$

$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = D = 4\pi\sigma$$

right side plate

$$\hat{n} = \hat{x} \quad D=0$$

$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = -D = 4\pi(-\sigma)$$

$$D = 4\pi\sigma \text{ as before}$$

$$\Rightarrow \vec{D} = 4\pi\sigma \hat{x}$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{4\pi\sigma}{\epsilon} \hat{x}}$$

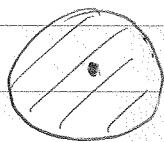
electric field reduced by factor $1/\epsilon$ as compared to capacitor with vacuum between plates

gong see 95

see Jackson section 4.4 for more interesting examples
- dielectric sphere in uniform applied E

see Jackson section (5.11) for an interesting magnetic b.c. problem
- spherical permeable shell in uniform applied B

point charge within a dielectric sphere



pt charge q at center of dielectric sphere of radius R , dielectric const ϵ

$$\vec{\nabla} \cdot \vec{D} = 4\pi q = \oint_S da \hat{n} \cdot \vec{D} = 4\pi Q_{\text{enc}}$$

From symmetry $\vec{D}(\vec{r}) = D(r) \hat{r}$

$$\oint_S da \hat{n} \cdot \vec{D} = 4\pi r^2 D(r) = 4\pi q$$

sphere of radius r $\rightarrow$ S

$$\vec{D} = \frac{q}{r^2} \hat{r} \quad \text{all } r$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{q}{\epsilon r^2} \hat{r} & r < R \\ \frac{q}{r^2} \hat{r} & r > R \end{cases}$$

can check that tangential component of $\vec{E}$ is continuous and normal component of $\vec{D}$ is continuous as there is no free σ at surface of dielectric.

normal component of $\vec{E}$ jumps by

$$\hat{n} \cdot (\vec{E}^{\text{above}} - \vec{E}^{\text{below}}) = \frac{q}{R^2} - \frac{q}{\epsilon R^2} = \frac{q}{R^2} \left(1 - \frac{1}{\epsilon}\right) = \frac{q}{R^2} \left(\frac{\epsilon - 1}{\epsilon}\right)$$

$$= \frac{q}{R^2} \left(\frac{4\pi \chi_e}{1 + 4\pi \chi_e} \right) = 4\pi \sigma_{\text{total}} = 4\pi \sigma_b$$

$$\Rightarrow \sigma_b = \frac{q}{4\pi R^2} \left(\frac{4\pi \chi_e}{1 + 4\pi \chi_e} \right) = \frac{q \chi_e}{R^2 \epsilon}$$

We can check this directly

$$\vec{P} = \chi_e \vec{E} = \frac{\chi_e}{\epsilon} \frac{q}{r^2} \hat{r}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{\chi_e}{\epsilon} q 4\pi \delta(r)$$

↑
bound charge at origin $q_b = -\frac{\chi_e}{\epsilon} 4\pi q$

total charge at origin is $q + q_b = q \left(1 - \frac{4\pi\chi_e}{\epsilon}\right)$

$$\epsilon = 1 + 4\pi\chi_e \quad = \frac{q}{\epsilon} \left(\frac{\epsilon - 4\pi\chi_e}{\epsilon}\right) = \frac{q}{\epsilon} \quad \text{screened charge}$$

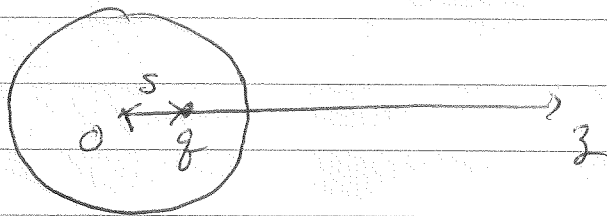
at surface,

$$\sigma_b = \hat{n} \cdot \vec{P} = \frac{\chi_e}{\epsilon} \frac{q}{R^2}$$

agrees with what we get from jump in $\hat{n} \cdot \vec{E}$.

Note: inside the dielectric the $\vec{E}$ field is that of the screened point charge $\frac{q}{\epsilon}$.
outside the dielectric $\vec{E}$ is just that of the free charge q . There is no evidence in $\vec{E}_{out}$ that the dielectric even exists!

Now consider same problem but q is off center



what is $\vec{E}$ inside + outside?

inside $\vec{\nabla} \cdot \vec{D} = 4\pi q$ where $\rho = q \delta(\vec{r} - s\hat{z})$

$$\vec{D} = \epsilon \vec{E} \Rightarrow \vec{\nabla} \cdot \vec{E} = 4\pi q / \epsilon$$

$$\vec{E} = -\vec{\nabla} \phi \Rightarrow \nabla^2 \phi = -\frac{4\pi q}{\epsilon} = -\frac{4\pi q}{\epsilon} \delta(\vec{r} - s\hat{z})$$

solution for ϕ will be of the form

$$\phi(\vec{r}) = \frac{q}{\epsilon |\vec{r} - s\hat{z}|} + F(\vec{r})$$

where 1st term is due to the point charge q/ϵ
 and 2nd term satisfies $\nabla^2 F = 0$ and will be
 chosen to get the correct behavior at the boundary
 of the dielectric

Since there is azimuthal symmetry about $\hat{z}$
 we can write

$$F(\vec{r}) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos \theta)$$

there are no $\frac{b_l}{r^{l+1}}$ terms since F should not
 diverge at the origin

So inside, $r < R$

$$\phi^{\text{in}}(\vec{r}) = \frac{q}{\epsilon |\vec{r} - s \hat{z}|} + \sum_{l=0}^{\infty} a_l r^l P_l(\cos \theta)$$

From our discussion of electric multipole expansion, we know we can write for $r > s$,

$$\frac{1}{|\vec{r} - s \hat{z}|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{s}{r}\right)^l P_l(\cos \theta)$$

So for $r > s$ (not true for $r < s$!)

$$\phi^{\text{in}}(\vec{r}) = \sum_{l=0}^{\infty} \left(\frac{q}{\epsilon r} \left(\frac{s}{r}\right)^l + a_l r^l \right) P_l(\cos \theta)$$

Outside the sphere there is no charge, so $\vec{\nabla} \cdot \vec{E} = 0$
or $\nabla^2 \phi = 0$

$$\Rightarrow \phi^{\text{out}}(\vec{r}) = \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos \theta)$$

there are no $a_l r^l$ terms since $\phi^{\text{out}} \rightarrow 0$ as $r \rightarrow \infty$

To determine the unknown a_l and b_l we use the boundary conditions at surface of dielectric at $r = R$

① Tangential component $\vec{E}$ is continuous

$$\vec{E} = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} = E_r \hat{r} + E_\theta \hat{\theta}$$

$\Rightarrow E_\theta$ is continuous at $r=R$

condition that E_θ is continuous is the same condition that ϕ is continuous (check this out for yourself if you are not sure)

$$\Rightarrow \phi^{\text{in}}(R, \theta) = \phi^{\text{out}}(R, \theta)$$

$$\text{as } \vec{E}^{\text{above}} - \vec{E}^{\text{below}} = 4\pi\sigma \hat{n}$$

$$\frac{q}{\epsilon R} \left(\frac{s}{R}\right)^l + a_l R^l = \frac{b_l}{R^{l+1}}$$

$$\Rightarrow \boxed{b_l = \frac{q}{\epsilon} s^l + a_l R^{2l+1}}$$

normal component $\vec{D}$ is continuous (since free surface charge $\sigma = 0$)

$$\vec{D} = \epsilon \vec{E}$$

$$\Rightarrow \epsilon E_r^{\text{in}} = E_r^{\text{out}}$$

$$-\epsilon \frac{\partial \phi^{\text{in}}}{\partial r} \bigg|_R = -\frac{\partial \phi^{\text{out}}}{\partial r} \bigg|_R$$

$$\Rightarrow \frac{(l+1)q}{R^2} \left(\frac{s}{R}\right)^l - l \epsilon a_l R^{l-1} = \frac{(l+1)b_l}{R^{l+2}}$$

$$g s^l - \frac{l}{l+1} \epsilon a_l R^{2l+1} = b_l$$

substitute in b_l from previous boundary condition

$$g s^l - \frac{l}{l+1} \epsilon a_l R^{2l+1} = \frac{g}{\epsilon} s^l + a_l R^{2l+1}$$

$$g s^l \left[1 - \frac{1}{\epsilon} \right] = a_l R^{2l+1} \left[1 + \frac{l}{l+1} \epsilon \right]$$

$$a_l = \frac{\frac{g s^l}{R^{2l+1}} \left[1 - \frac{1}{\epsilon} \right]}{\left[1 + \left(\frac{l}{l+1} \right) \epsilon \right]}$$

$$b_l = \frac{g}{\epsilon} s^l + a_l R^{2l+1}$$

$$= \frac{g}{\epsilon} s^l + g s^l \frac{\left[1 - \frac{1}{\epsilon} \right]}{\left[1 + \left(\frac{l}{l+1} \right) \epsilon \right]}$$

$$b_l = \frac{g s^l}{\epsilon} \left\{ 1 + \frac{\epsilon - 1}{1 + \left(\frac{l}{l+1} \right) \epsilon} \right\}$$

$$= \frac{g s^l}{\epsilon} \left[\frac{\epsilon \left(1 + \frac{l}{l+1} \right)}{1 + \left(\frac{l}{l+1} \right) \epsilon} \right]$$

$$b_l = g s^l \left[\frac{1 + \left(\frac{l}{l+1} \right)}{1 + \left(\frac{l}{l+1} \right) \epsilon} \right]$$

check the result:

as $s \rightarrow 0$, should recover previous answer

for $s=0$, $a_l = b_l = 0$ for all $l \neq 0$

$$a_0 = \frac{q}{R} \left[1 - \frac{1}{\epsilon} \right]$$

$$b_0 = q$$

$$s_0 \quad \phi^{\text{in}}(\vec{r}) = \frac{q}{\epsilon r} + \frac{q}{R} \left[1 - \frac{1}{\epsilon} \right]$$

$$\vec{E}^{\text{in}} = -\vec{\nabla} \phi^{\text{in}} = \frac{q}{\epsilon r^2} \hat{r} \quad \text{as before}$$

$$\phi^{\text{out}}(\vec{r}) = \frac{q}{r}$$

$$\vec{E}^{\text{out}} = -\vec{\nabla} \phi^{\text{out}} = \frac{q}{r^2} \hat{r} \quad \text{as before}$$

note: the constant that is the 2nd term in ϕ^{in}
is just what is needed to make ϕ continuous at $r=R$

another check:

let $\epsilon \rightarrow \infty$ this models a conductor!

again one finds $a_l = b_l = 0$ for all $l \neq 0$

$$a_0 = \frac{q}{R}$$

$$b_0 = q$$

$$\phi^{\text{in}}(\vec{r}) = \frac{\sum q(S) \rho_l}{\epsilon r^l} + \frac{q}{R} \rightarrow \frac{q}{R} \text{ as } \epsilon \rightarrow \infty$$

$\Rightarrow E^{\text{in}}(\vec{r}) = 0$ as ϕ^{in} is a constant.

$$\phi^{\text{out}}(\vec{r}) = \frac{q}{r} \Rightarrow \vec{E}^{\text{out}} = \frac{q}{r^2} \hat{r}$$

field outside is like point charge q at the origin,
independent of where q is inside the sphere.
This is the correct behavior of a conductor.

The mobile charges in the conductor completely
screen the q inside, and leave a uniform
surface charge $\sigma_b = \frac{q}{4\pi R^2}$ on the surface.