# Green's theorem, Uniqueness, Green function part II

We want to show that the boundary value problem we described is well posed – ie there is a unique solution. We start by deriving Greens theorems

Consider $\int_V d^3r\, \vec{\nabla}\cdot\vec{A} = \oint_S da\, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ $\phi, \psi$ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\,\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\phi\frac{\partial\psi}{\partial n}$$ Green's 1st identity

let $\phi \leftrightarrow \psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)$$ Green's 2nd identity

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec{r}-\vec{r}'|}$, $\vec{r}'$ is integration variable, $\phi$ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. Use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$\int_V d^3r'\left[\phi(r')\left[-4\pi\,\delta(r-r')\right] - \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-4\pi\rho(\vec{r}')\right)\right]$$

$$= \oint_S da'\left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

Note: if $\vec{r}$ lies outside the volume V, then

$$(**) \qquad 0 = \int_V d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \oint \frac{da'}{4\pi}\left[\underbrace{\frac{1}{|\vec{r}-\vec{r}\,'|}\frac{\partial\phi}{\partial n'}}_{\text{potential from a surface charge density } \sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}} - \underbrace{\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)}_{\text{potential from a surface dipole layer of dipole strength density } \frac{\phi}{4\pi}}\right]$$

From (*), if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int d^3r' \, \rho(\vec{r}\,')/|\vec{r}-\vec{r}\,'|$

(*) gives the generalization of Coulombs law to a system with a finite boundary

For a <u>charge free</u> volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify <u>both</u> $\phi$ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting $\phi$ from (*) would not in general obey Laplace's equation $\nabla^2\phi = 0$, nor would (**) vanish.

Specifying both $\phi$ and $\frac{\partial\phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's equ, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

## Uniqueness

If we have a system of charges in vol $V$, and *either* the potential $\phi$, *or* its normal derivative $\frac{\partial\phi}{\partial n}$, is specified on the surfaces of $V$, then there is a unique solution to Poisson's equation inside $V$. Specifying $\phi$ is known as Dirichlet boundary conditions. Specifying $\frac{\partial\phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions $\phi_1$ and $\phi_2$, *both* with $-\nabla^2\phi = 4\pi\rho$ inside $V$, and obeying specified b.c. on surface of $V$.

Define $u = \phi_2 - \phi_1 \rightarrow \nabla^2 u = 0$ inside $V$

and $u = 0$ on surface $S$ - for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface $S$ - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \,(\underbrace{u\nabla^2 u}_{\substack{\shortparallel \\ 0 \\ \text{as } \nabla^2 u = 0}} + \vec\nabla u\cdot\vec\nabla u) = \oint_S da\, \underbrace{u\frac{\partial u}{\partial n}}_{\substack{\shortparallel \\ 0 \\ \text{as } u \text{ or } \frac{\partial u}{\partial n} = 0}}$$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} u|^2 = 0 \qquad \Rightarrow \vec{\nabla} u = 0$$

$$\Rightarrow u = \text{const}$$

For Dirichlet b.c., $u = 0$ on surface $S$, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., $\phi_1$ and $\phi_2$ differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

If boundary ~~solution~~ surface $S$ consists of several disjoint pieces, then solution is unique if specify $\phi$ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both $\phi$ and $\frac{\partial \phi}{\partial n}$ specified on the same surface $S$ (Cauchy b.c.) does not in general exist, since specifying either $\phi$ or $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.

# Green's function – part II

Greens $2^{nd}$ identity

$$\int_V d^3r' \left( \phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left( \phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2\phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r},\vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

we saw one solution of above is $G(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|}$
but a more general solution is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

where $\nabla'^2 F(\vec{r},\vec{r}') = 0$, for $\vec{r}'$ in volume $V$
we will choose $F(\vec{r},\vec{r}')$ to simplify solution of $\phi$

$$\Rightarrow \int_V d^3r' \left( \phi(\vec{r}')\, \nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\nabla'^2\phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left( \phi(r') \left[-4\pi\delta(\vec{r}-\vec{r}')\right] - G(\vec{r},\vec{r}') \left[-4\pi\rho(\vec{r}')\right] \right)$$

$$= -4\pi\,\phi(\vec{r}) + 4\pi \int_V d^3r'\, G(\vec{r},\vec{r}')\,\rho(\vec{r}')$$

$$= \oint_S da' \left( \phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r'\, G(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint_S \frac{da'}{4\pi}\left( G(\vec{r},\vec{r}')\frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(r')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right)$$

Consider <u>Dirichlet boundary problem</u>. If we can choose $F(\vec{r},\vec{r}')$ such that $G_D(\vec{r},\vec{r}')=0$ for $\vec{r}'$ on the boundary surface $S$, then above simplifies to

$$\boxed{\phi(\vec{r}) = \int_V d^3r'\, G_D(\vec{r},\vec{r}')\, \rho(\vec{r}') - \oint_S \frac{da'}{4\pi}\, \phi(r')\, \frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}}$$

Since $\rho(r)$ is specified in $V$, and $\phi(r)$ is specified on $S$, above then gives desired solution ~~of~~ for $\phi(r)$ inside volume $V$.

Finding $G_D$ is therefore equivalent to finding an $F(r,r')$ such that ~~[illegible]~~ $\nabla'^2 F(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ in $V$ (solves Laplace eqn) and

$F(\vec{r},\vec{r}') = \dfrac{-1}{|\vec{r}-\vec{r}'|}$ for $\vec{r}'$ on boundary surface $S'$

Always exists unique solution for $F$

Next consider <u>Neumann boundary problem</u>.

One might think to find $F(\vec{r},\vec{r}')$ such that $\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\int_V \nabla'^2 G(\vec{r},\vec{r}')\, d^3r' = \int \vec{\nabla}' \cdot \vec{\nabla}' G(\vec{r},\vec{r}')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}') \cdot \hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, da' = -4\pi$$ since $\nabla'^2 G = -4\pi\delta(r-r')$

So we can't have $\frac{\partial G}{\partial n'} = 0$ for $\vec{r}'$ on $S$

Simplest choice is then $\frac{\partial G_N(r,r')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}'$ on $S$

area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}')\, \frac{\partial \phi(\vec{r}')}{\partial n'}$$

$$- \oint \frac{da'}{4\pi}\, \phi(r') \left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}')\, \rho(\vec{r}') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}')\, \frac{\partial \phi(\vec{r}')}{\partial n'}\right]$$

$$+ \langle \phi \rangle_S$$

constant = average value of $\phi$ on surface $S$.

Since $\rho(r)$ is specified in $V$ and $\frac{\partial \phi}{\partial n}$ is specified on $S$

above gives solution $\phi(r)$ in $V$ within additive constant $\langle \phi \rangle_S$

Since $\vec{E} = -\vec{\nabla}\phi$, the const $\langle \phi \rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

and $$\frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While $G_D$ and $G_N$ always exist in principle, they depend in detail on the shape of the surface $S'$ and are difficult to find except for simple geometries

In preceeding we defined $G$ by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface $S'$, $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of $G$ above is obtained by noting that one can prove the symmetry property

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10)