Separation of Variables

If the system has a rectangular boundary, $\underbrace{}_{}$ and contains <u>no</u> charge, we can look for solutions to $\nabla^2\phi = 0$ of the form

$$\phi(\vec{r}) = X(x)\, Y(y)\, Z(z)$$

product of three function each of one variable only

$$\nabla^2\phi = 0 \Rightarrow \frac{1}{\phi}\nabla^2\phi = 0$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0$$

The only way this can happen for <u>all</u> values of x, y, z is if each of the three terms is a constant, call them a^2, b^2, c^2

$$\frac{1}{X}\frac{d^2X}{dx^2} = a^2 \quad \Rightarrow \quad X(x) = A_1 e^{-ax} + A_2 e^{ax}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = b^2 \qquad Y(y) = B_1 e^{-by} + B_2 e^{by}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = c^2 \qquad Z(z) = C_1 e^{-cz} + C_2 e^{cz}$$

with $a^2 + b^2 + c^2 = 0$

$\Rightarrow$ at least one of the a^2, b^2, c^2 is < 0

$\Rightarrow$ at least one of the a, b, c is <u>imaginary</u>.

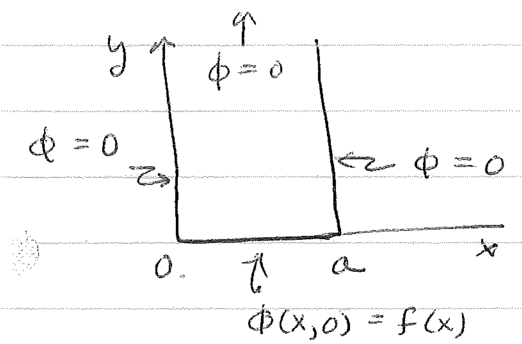
Above is one particular solution. But there are many solutions, each with different a, b, c, but all obeying the constraint $a^2+b^2+c^2=0$. The General solution is a superposition of these

$$\phi(x,y,z) = \sum_i \left(A_{1i}\, e^{-a_i x} + A_{2i}\, e^{a_i x}\right)\left(B_{1i}\, e^{-b_i y} + B_{2i}\, e^{b_i y}\right)\left(C_{1i}\, e^{-c_i z} + C_{2i}\, e^{c_i z}\right)$$

$$a_i^2 + b_i^2 + c_i^2 = 0$$

<u>Example</u>

Consider a channel shaped as below - infinite along z



$\phi(0,y) = 0$

$\phi(a,y) = 0$

$\phi(x,y) = 0$ as $y \to \infty$

$\phi(x,0) = f(x)$ specified function

solution is independent of z $\Rightarrow$

$$\phi(x,y) = \sum_i \left(A_{1i}\, e^{-a_i x} + A_{2i}\, e^{a_i x}\right)\left(B_{1i}\, e^{-b_i y} + B_{2i}\, e^{b_i y}\right)$$

$$a_i^2 + b_i^2 = 0$$

we will see that the correct thing is to choose a imaginary

let $a_i = i\alpha_i$

$b_i = \alpha_i$

$$\phi(x,y) = \sum_i \left(A_i \cos\alpha_i x + B_i \sin\alpha_i x\right)\left(C_i\, e^{-\alpha_i y} + D_i\, e^{\alpha_i y}\right)$$

where $A_i = (A_{1i} + A_{2i})$ $\quad C_i = B_{1i}$

$B_i = i(A_{1i} - A_{2i})$ $\quad D_i = B_{2i}$

Now $\phi(x,y) \to 0$ as $y \to \infty$ for all x $\Rightarrow$ $\boxed{D_i = 0}$

$$\Rightarrow \quad \phi(x,y) = \sum_i \left[A_i' \cos\alpha_i x + B_i' \sin\alpha_i x\right] e^{-\alpha_i y}$$

where $A_i' = A_i C_i$, $B_i' = B_i C_i$

$$\phi(0,y) = 0 \Rightarrow \sum_i A_i' e^{-\alpha_i y} = 0 \text{ all } y \Rightarrow \boxed{A_i' = 0}$$

$$\Rightarrow \phi(x,y) = \sum_i B_i' \sin(\alpha_i x) e^{-\alpha_i y}$$

$$\phi(a,y) = 0 \Rightarrow \sum_i B_i' \sin(\alpha_i a) e^{-\alpha_i y} = 0 \text{ all } y$$

$$\Rightarrow \sin(\alpha_i a) = 0 \quad \text{or} \quad \alpha_i a = n\pi$$

$\alpha_i = \frac{n\pi}{a}$ integer $n \geq 1$

$$\Rightarrow \boxed{\phi(x,y) = \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}}$$

Finally

$$\phi(x,0) = f(x) \Rightarrow \underbrace{\sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) = f(x)}$$

This is just the Fourier series for $f(x)$!

$$\boxed{B_n' = \frac{2}{a}\int_0^a dx\, f(x) \sin\left(\frac{n\pi x}{a}\right)}$$

We have thus determined all unknown coefficients and found the solution!

See Jackson 2-8 if Fourier series needs review

Recall orthogonality : $\frac{2}{a}\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

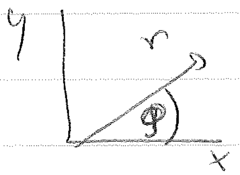
For $f(x) = \Phi_0$ a constant,

$$B_n' = \frac{2}{a}\Phi_0 \int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right) = \frac{2\Phi_0}{a}\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2\Phi_0}{n\pi}\left(1-\cos n\pi\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{4\Phi_0}{n\pi} & n \text{ odd} \end{cases}$$

Polar Coordinates — still translationally invariant along z — so two dimensional

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$



assume $\phi(r,\varphi) = R(r)\,\Phi(\varphi)$

$$\frac{r^2\nabla^2\phi}{\phi} = \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0$$

each term must be a constant

$$\Rightarrow \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \nu^2, \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\nu^2$$

Solutions are

$$\left.\begin{aligned} R(r) &= a r^{\nu} + b r^{-\nu} \\ \Phi(\varphi) &= A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{aligned}\right\} \quad \nu \neq 0$$

$$\left.\begin{aligned} R(r) &= a_0 + b_0 \ln r \\ \Phi(\varphi) &= A_0 + B_0\varphi \end{aligned}\right\} \quad \nu = 0$$

$\cancel{\Phi(r,\varphi) = (a_0 + b_0\ln r)(A_0 + B_0\varphi) +}$

If φ can take its entire range from 0 to 2π ~~then~~ (such as problem in which ϕ is specified on the surface of a cylinder) then periodicity in $\varphi \to \varphi + 2\pi$ requires $B_0 = 0$ and $\nu =$ integer n

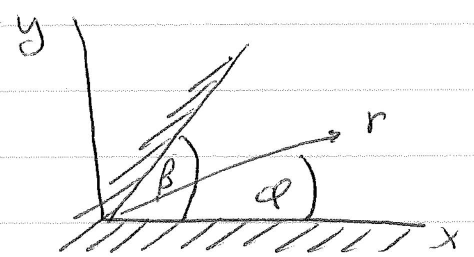
$$\phi = a_0 + b_0 \ln r + \sum_{n=1}^{\infty}\Big[r^n\left(A_n\cos(n\varphi) + B_n\sin(n\varphi)\right) + r^{-n}\left(C_n\cos(n\varphi) + D_n\sin(n\varphi)\right)\Big]$$

or reparameterizing

$$\phi(r,\varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[a_n r^n \sin(n\varphi + \alpha_n) + b_n r^{-n} \sin(n\varphi + \beta_n) \right]$$

If the region where there is no charge includes $r=0$, then all $b_n = 0$ since ϕ should not diverge at the origin. If $r=0$ is excluded from the region, then the b_n need not be zero. The case $b_0 \neq 0$ corresponds to a line charge λ along the z axis.

Consider the case where φ has a restricted range, for example a wedge shaped opening of angle β.



shaded region is conductor

$0 \leq \varphi \leq \beta$

ϕ is constant in conductor

$\Rightarrow$ boundary conditions

$$\begin{cases} \phi(r,0) = \phi_0 \\ \phi(r,\beta) = \phi_0 \end{cases}$$

The general solution is the linear combination

$$\phi(r,\varphi) = (a_0 + b_0 \ln r)(A_0 + B_0 \varphi) + \sum_{\nu > 0} \left(a_\nu r^\nu + b_\nu r^{-\nu}\right)\left(A_\nu \cos(\nu\varphi) + B_\nu \sin(\nu\varphi)\right)$$

(1) The condition $\phi(r,0) = \phi_0$ a constant indep of r then requires

$$b_0 = 0 \quad , \quad A_\nu = 0 \quad \text{all } \nu$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} (a_\nu r^\nu + b_\nu r^{-\nu}) B_\nu \sin(\nu\varphi)$$

(2) Since ϕ should be continuous as one approaches the conducting surface, and $\phi = \phi_0$ is a finite constant on the conducting surface, then ϕ cannot diverge as one approaches the origin $r=0$ along any fixed angle φ. This requires

$$b_\nu = 0 \quad \text{all } \nu$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} a_\nu r^\nu \sin(\nu\varphi)$$

(3) The condition $\phi(r,\beta) = \phi_0$ a constant independent of r then requires

$$\sin(\nu\beta) = 0 \quad \Rightarrow \quad \nu = \frac{n\pi}{\beta} \quad , \quad n \text{ integer} \geq 1$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

(4) as ϕ must approach the constant ϕ_0 as $r \to 0$ along any fixed angle φ, we therefore must have

$$B_0 = 0 \quad , \quad a_0 A_0 = \phi_0$$

So finally we have

$$\phi(r,\varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

We still have all the unknown a_n! These depend on how $\phi(r,\varphi)$ behaves as $r \to \infty$ (we can't make the choice here that $\phi \to 0$ as $r \to \infty$) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small r. In this case, the leading term in the above series expansion for ϕ is the $n=1$ term, as it vanishes most slowly as $r \to 0$.

$$\phi(r,\varphi) \simeq \phi_0 + a_1 r^{\pi/\beta} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

The electric field is

$$E_r(r,\varphi) = -\frac{\partial\phi}{\partial r} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

$$E_\varphi(r,\varphi) = -\frac{1}{r}\frac{\partial\phi}{\partial\varphi} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi\varphi}{\beta}\right)$$

$$\Rightarrow \boxed{E \sim r^{\frac{\pi}{\beta}-1}}$$

Induced surface charge given by $4\pi\sigma = \vec{E}\cdot\hat{n}$

for surface at $\varphi = 0$, $\hat{n} = \hat{\varphi}$

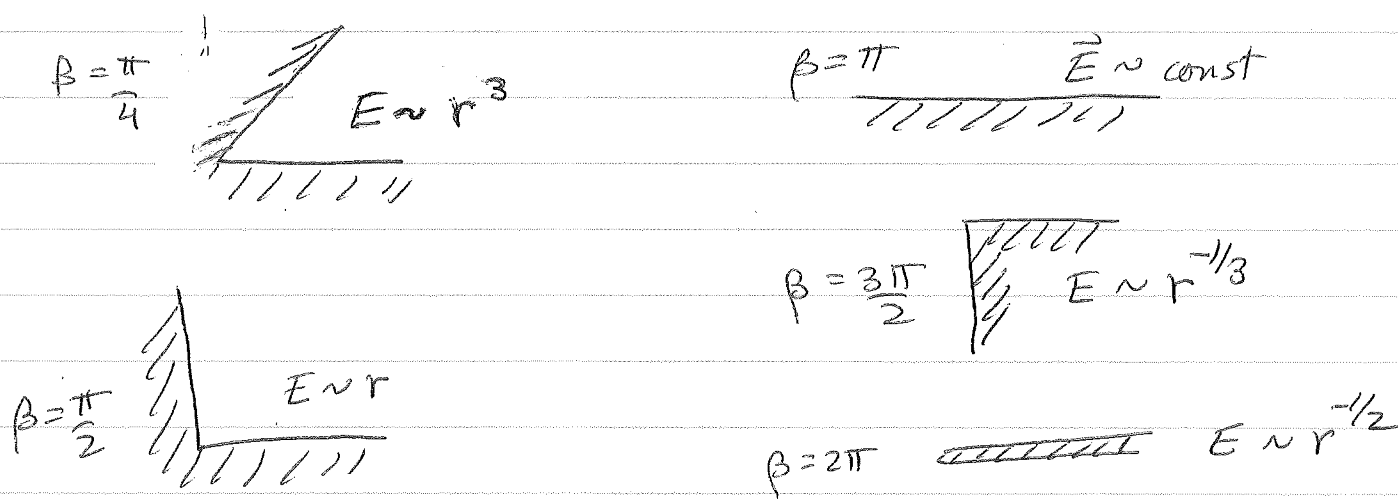
for surface at $\varphi = \beta$, $\hat{n} = -\hat{\varphi}$

$$\sigma(r, \varphi = 0) = \frac{E_\varphi(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

$$\sigma(r, \varphi = \beta) = \frac{-E_\varphi(r, \beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

For $\frac{\pi}{\beta} > 1$, ie $\beta < \pi$, $\vec{E}$ and σ vanish as approach the origin.

For $\frac{\pi}{\beta} < 1$, ie $\beta > \pi$, $\vec{E}$ ad σ diverge as approach the origin



E diverges at an "extrnal" corner

E vanishes at an "internal" corner

Remember, the above exaples all had translational symmetry along z, so the "corners" above are really infinitely long straight "edges".

Spherical Coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$r^2\nabla^2\phi = \Theta\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{r^2\sin^2\theta\,\nabla^2\phi}{\phi} = \underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\substack{\text{depends only on } r \text{ and } \theta \\ = -\text{const}}} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\substack{\text{depends only on} \\ \varphi \\ = \text{const}}} = 0$$

take $\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\varphi^2} = -m^2$

$\Rightarrow \boxed{\Phi = e^{\pm i m\varphi}}$ m integer for 2π periodicity in φ

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\substack{\text{depends only on } r \\ = \text{const}}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\substack{\text{depends only on } \theta \\ = -\text{const}}} = 0$$