Electromagnetic Waves in a vacuum

No sources $\vec{J}=0$, $\rho=0$

1) $\vec{\nabla}\cdot\vec{E}=0$ 3) $\vec{\nabla}\cdot\vec{B}=0$

2) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{c\,\partial t}$ 4) $\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{0 \text{ by } (1)} - \nabla^2\vec{E} = -\frac{\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B})$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{c\,\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Similarly

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

} wave equation
wave speed is c.

Note: in MKS units, above wave equation looks like

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic wave

Harmonic

Plane waves

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
complex exponential form

$\vec{k}$ is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \frac{2\pi}{|k|}$ is wavelength

$\left.\begin{array}{l}|\vec{E}_k| \\ |\vec{B}_k|\end{array}\right\}$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space with period λ

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E}=0 \Longrightarrow \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$
$$\Rightarrow \vec{E}_k\cdot\vec{k}=0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}_k\cdot\vec{k}=0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \operatorname{Re}\left[\vec{\nabla}\times \vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[\frac{1}{c}\vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \operatorname{Re}\left[-\vec{B}_k\times\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \operatorname{Re}\left[i\vec{k}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \vec{k}\times\vec{B}_k = -\frac{\omega}{c}\vec{E}_k$$

$$\vec{k}\times\vec{k}\times\vec{B}_k = -k^2\vec{B}_k = -\frac{\omega}{c}\vec{k}\times\vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2}\vec{k}\times\vec{E}_k} = \frac{\omega}{ck}\hat{k}\times\vec{E}_k$$

Finally

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \operatorname{Re}\left[\vec{E}_k\nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \frac{\vec{E}_k}{c^2}\frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \operatorname{Re}\left[\vec{E}_k(-k^2)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\omega^2}{c^2}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}$$

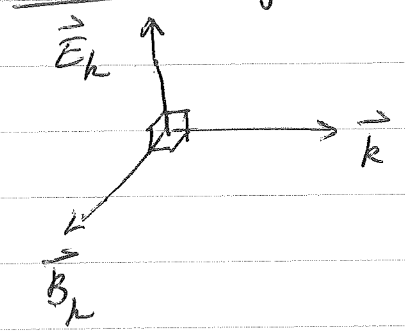
$$\boxed{\omega = \pm kc}$$ <u>dispersion relation</u>

insert in above

$$\vec{B}_k = \hat{k}\times\vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow |\vec{B}_k| = |\vec{E}_k|$$

Summary



$$\left.\begin{array}{l}\vec{E}_k \perp \vec{k} \\ \vec{B}_k \perp \vec{k}\end{array}\right\} \text{"transverse" polarization}$$

$$\vec{B}_k = \hat{k} \times \vec{E}_k$$

$$\omega^2 = c^2 k^2$$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge q is

$$q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above harmonic plane waves

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ Fourier transform

$\vec{E}(\vec{r},t)$ is real $\Rightarrow$ $\vec{E}_k^{\,*} = \vec{E}_{-k}$

For dispersion relation $\omega^2 = c^2k^2$ we can write

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} - \vec{v}t)$$

where $\vec{v} = c\hat{k}$ is velocity of wave.
If we only combine waves traveling in same direction $\hat{k}$, then

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i\vec{k}\cdot(\vec{r} - \vec{v}t)} = \vec{E}(\vec{r} - \vec{v}t, 0)$$

The general plane wave solution of wave equation always has this property

$$\vec{E}(r,t) = \vec{E}(\vec{r} - \vec{v}t, 0)$$

If know $\vec{E}$ at $t=0$, then know $\vec{E}$ at all times t

Energy & momentum in EM wave

$$\vec{E} = Re\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$\vec{B} = Re\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \hat{k}\times\vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$

for real $\vec{E}_k$

energy density $\quad u = \frac{1}{8\pi}\left(E^2 + B^2\right)$

$$= \frac{1}{8\pi}\left[E_k^2 + E_k^2\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current $\quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$$= \frac{c}{4\pi}\left[\vec{E}_k \times (\hat{k}\times\vec{E}_k)\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi}\hat{k} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c\, u\, \hat{k}$$

momentum density $\quad \vec{\Pi} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$

$u = c|\vec{\Pi}|$ — energy momentum relation of photons!

For visible light $\lambda \sim 5\times 10^{-7}\,\text{m} \sim 5000\,\text{Å}$

$$T = \frac{\lambda}{c} = 1.6\times 10^{-15}\ \text{sec}$$

most classical measurements on microscopic scales $t \gg T$, $\ell \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\, u = \frac{1}{8\pi} E_k^2 \qquad \text{as } \langle \cos^2\theta \rangle = 1/2$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal $\hat{n}$

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources, $\rho = 0$, $\vec{j} = 0$.
("free" charge and current vanishes)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

linear materials

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

if μ and ϵ were simply constants then the above would become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

Then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t}\right)$$

$$= -\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu\epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minor changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$ dispersion relation changed by constant factor

$$\vec{E}_k \perp \vec{k}$$
$$\vec{B}_k \perp \vec{k}$$

$$i\vec{k} \times \vec{E}_k = \frac{i\omega}{c} \vec{B}_k$$

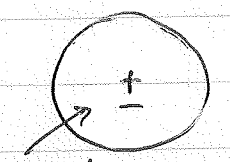
$$\frac{c|\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \quad \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E}_k = \vec{B}_k \qquad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu\epsilon}} < c$

In general however things are <u>much more complicated</u> because ϵ cannot be viewed as a constant when considering time varying behavior!

Time dependent polarizeability of an atom



electron cloud

If displace center of electron cloud by a distance $\vec{r}$, there is a restoring force $\vec{F}_{rest} = \dfrac{-e^2 \vec{r}}{4\pi R^3} \equiv -m\omega_0^2 \vec{r}$

(m: electron mass; ω_0: resonant frequency)

Also, in general there will be a damping force

$$\vec{F}_{damp} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = \frac{-e\vec{E}(t)}{m}$$

assuming $\vec{E}$ is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2\vec{r}_0 - i\omega\gamma\vec{r}_0 + \omega_0^2\vec{r}_0 = \frac{-e\vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_0$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) E_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$ freq dependent polarizability

Since α is complex the polarization does not in general oscillate in phase with $\vec{E}$.

If $\alpha(\omega) = |\alpha| e^{i\delta}$ δ is phase of complex α

$\alpha = \alpha_1 + i\alpha_2$ then $|\alpha| = \sqrt{\alpha_1^2 + \alpha_2^2}$ $\tan\delta = \alpha_2/\alpha_1$

$$\vec{p}(t) = \alpha(\omega)\vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

phase shifted by δ

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} dt'\, E(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \tilde{\alpha}(t-t')$$

Fourier transf of $\alpha(\omega)$

$\vec{p}$ at time t is due to $\vec{E}$ at all times t'

non local in time

$\chi(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\chi}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

do by contour integration

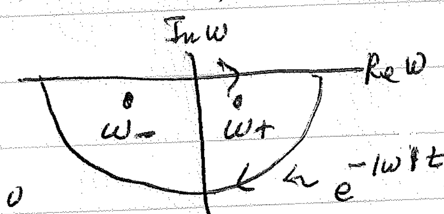
$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

poles at $\omega_\pm$ are in lower half complex plane.

for $t > 0$, close contour in lower half plane

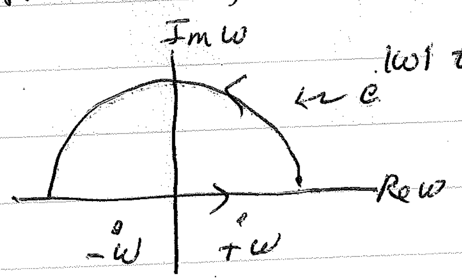
$t > 0$



← $e^{-|\omega|t}$ here vanishes for $t>0$

contour encloses poles + get contribution

for $t < 0$, close contour in upper half plane



← $e^{|\omega|t}$ vanishes for $t<0$

contour encloses no poles ⇒ integral vanishes

$\chi(t) = 0$ for $t < 0$

<u>causal response</u>! No polarization <u>until electric field turns on</u>

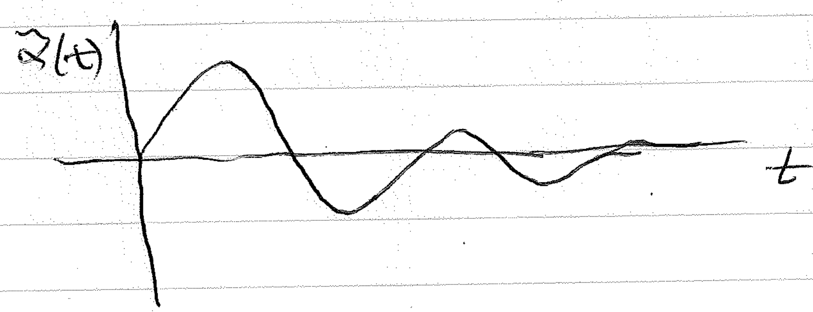
For $t > 0$

$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$= \underbrace{(-2\pi i)}_{\text{from residue theorem}} \frac{e^2}{m} \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

$$= \frac{i e^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \dfrac{e^2}{m} \dfrac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



damped oscillation

Polarization density $\vec{P}_\omega = 4\pi \chi(\omega) \vec{E}_\omega$ for harmonic oscillation

$\chi(\omega) \approx \underbrace{n}_{\text{atom density}} \alpha(\omega)$ for dilute system

can use Clausius - Mossotti correction for denser materials

$\Rightarrow \vec{D}_\omega = \underbrace{\varepsilon(\omega)}_{\text{freq dependent}} \vec{E}_\omega$ $\qquad \varepsilon(\omega) = 1 + 4\pi \chi(\omega)$