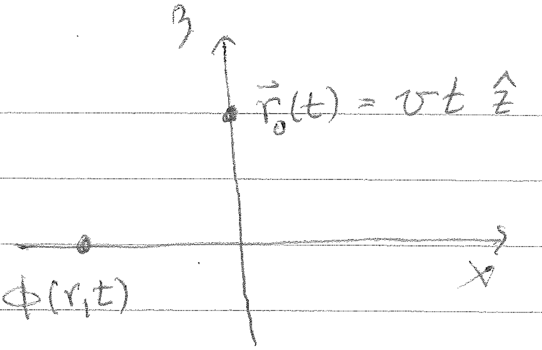


~~at x potential from q at $vt\hat{z}$~~

potential at pt $\vec{r}$ in xy plane at time t, when charge is at $\vec{r}_0 = vt\,\hat{z}$, looks almost like static Coulomb potential, which would be $\dfrac{q}{\sqrt{r^2+v^2t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2t^2+\left(\frac{r}{\gamma}\right)^2}},$$

looks like the transverse direction has contracted by a factor γ!

Such considerations led Lorentz to discover the Lorentz transformation, before Einstein proposed his theory of special relativity

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \frac{q\,(\vec{r}-\vec{v}t)\,\gamma^{-2}}{\left[(\vec{r}-\vec{v}t)^2 + \left(\frac{\vec{r}\cdot\vec{v}}{c}\right)^2 - \left(\frac{rv}{c}\right)^2\right]^{3/2}}$$

$$\simeq q\,\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3} \quad \text{as } \frac{v}{c}\to 0$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\vec{v}}{c^2}\times\vec{E}$$

$$= q\vec{v}\times\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3} \quad \text{as } \frac{v}{c}\to 0$$

looks like Biot-Savart Law with $\vec{J} = q\vec{v}\,\delta(\vec{r}-\vec{v}t)$

Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r'\,dt' \; \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\; \vec{j}(\vec{r}',t')$$

for pure harmonic oscillation in current

$$\vec{j}(\vec{r},t) = \mathrm{Re}\left\{ \vec{j}_\omega(\vec{r})\, e^{-i\omega t} \right\}$$

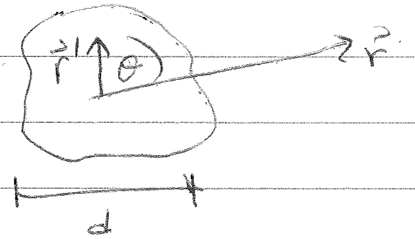
$$\Rightarrow \quad \vec{A}(r,t) = \mathrm{Re}\left\{ \vec{A}_\omega(\vec{r})\, e^{-i\omega t} \right\}$$

$$\Rightarrow \quad \vec{A}_\omega(\vec{r})e^{-i\omega t} = \frac{1}{c}\int d^3r' \; \vec{j}_\omega(\vec{r}')\, e^{-i\omega t}\, \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

doing $\int dt'$ by using the δ-function

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r' \; \vec{j}_\omega(\vec{r}')\, \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume source is localized, ie $\vec{j}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$



Approx ①

for $r \gg d$, far from sources

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta}$$

$$\simeq r\left(1 - \frac{r'}{r}\cos\theta\right)$$

$$\simeq r - \vec{r}'\cdot\hat{r} + O\left(\frac{r'}{r}\right)^2$$

$$\hat{r} \equiv \frac{\vec{r}}{r}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}\,') \, e^{ik(r-\vec{r}\,'\cdot\hat{r})}}{r-\vec{r}\,'\cdot\hat{r}} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}\,') \, e^{-ik\vec{r}\,'\cdot\hat{r}}}{1-\frac{\hat{r}\cdot\vec{r}\,'}{r}}$$

$$\approx \frac{e^{ikr}}{cr}\int d^3r' \, \vec{J}_\omega(\vec{r}\,') \, e^{-ik\hat{r}\cdot\vec{r}\,'} \left(1+\frac{\hat{r}\cdot\vec{r}\,'}{r}\right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \, \vec{J}_\omega(\vec{r}\,') \ldots$ term will determine the angular dependence of the radiation.

Approx ② $\lambda \gg d$ long wave length approx
or $kd \ll 1 \Rightarrow \frac{\omega}{c} d \ll 1$ or $\frac{d}{\tau} \ll c$
where τ is period of oscillation.
Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\Rightarrow \lambda \gg d$ is a non-relativistic approximation

$kd \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}\,'} \approx 1 - ik\hat{r}\cdot\vec{r}\,' +$ higher orders

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{cr}\int d^3r'\,\vec{j}_\omega(\vec{r}\,')\,(1 - ik\hat{r}\cdot\vec{r}\,')\left(1 + \frac{\hat{r}\cdot\vec{r}\,'}{r}\right)$$

$$= \frac{e^{ikr}}{cr}\int d^3r'\,\vec{j}_\omega(\vec{r}\,')\left[1 + \hat{r}\cdot\vec{r}\,'\left(\frac{1}{r} - ik\right)\right]$$

+ higher order in $\frac{d}{r}$ or kd

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r}\left[\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right]$$

where $\vec{I}_1 \equiv \frac{1}{c}\int d^3r'\,\vec{j}_\omega(\vec{r}\,')$

$\vec{I}_2 \equiv \frac{1}{c}\int d^3r'\,\hat{r}\cdot\vec{r}\,'\,\vec{j}_\omega(\vec{r}\,')$

Consider first $\vec{I}_1$ ith component ($\vec{I}_1$ vanishes in statics)

$$\int d^3r\,j_i(\vec{r}) = -\int d^3r\,r_i\,\vec{\nabla}\cdot\vec{j} \quad \text{integration by parts}$$

since $j_i = (\vec{\nabla}r_i)\cdot\vec{j} = \vec{\nabla}\cdot(r_i\vec{j}) - r_i\vec{\nabla}\cdot\vec{j}$

$$= \int d^3r\,r_i\frac{\partial\rho}{\partial t} \quad \text{as } \vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

$$\int d^3r\,j_i(\vec{r}) = -i\omega\int d^3r\,r_i\,\rho_\omega(\vec{r})$$

$$\Rightarrow \vec{I}_1 = -\frac{i\omega}{c}\int d^3r\,\vec{r}\rho_\omega(\vec{r}) = -\frac{i\omega}{c}\vec{p}_\omega$$

electric dipole moment

Electric dipole approximation from $\vec{I}_1$

$$\boxed{\vec{A}_{E1}(\vec{r}) = \frac{e^{ikr}}{r}\left(\frac{-i\omega\vec{p}_\omega}{c}\right) = -i\,\vec{p}_\omega\,\frac{k\,e^{ikr}}{r}} \qquad \omega = ck$$

Consider $\vec{I}_2$

$$\vec{I}_2 = \frac{1}{c}\int d^3r'\ \hat{r}\cdot\vec{r}'\,\vec{J}_\omega(\vec{r}') = \frac{1}{c}\hat{r}\cdot\int d^3r'\ \underbrace{\vec{r}'\,\vec{J}_\omega(\vec{r}')}_{\text{tensor}}$$

we saw this tensor ~~earlier~~ when we did the magnetic dipole approx, and when we derived the macroscopic Maxwell equations

$$\int d^3\vec{r}'\ \vec{r}'\,\vec{J}_\omega(\vec{r}') = -\int d^3r'\ \vec{J}_\omega(\vec{r}')\,\vec{r}' - \int d^3r'\ \vec{r}'\vec{r}'(\vec{\nabla}'\cdot\vec{J}_\omega(r'))$$

$$= \frac{1}{2}\int d^3r'\left[\vec{r}'\vec{J}_\omega - \vec{J}_\omega\vec{r}'\right] - \frac{1}{2}\int d^3r'\ i\omega\,\vec{r}'\vec{r}'\rho_\omega$$

using $\vec{\nabla}'\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

$$\vec{I}_2 = \frac{1}{2c}\int d^3r'\left[(\hat{r}\cdot\vec{r}')\vec{J}_\omega - (\hat{r}\cdot\vec{J}_\omega)\vec{r}'\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'(\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\frac{1}{2c}\int d^3r'\left[\hat{r}\times(\vec{r}'\times\vec{J}_\omega)\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'\ (\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}'_\omega$$

where $\vec{m}_\omega = \frac{1}{2c}\int d^3r'\ \vec{r}'\times\vec{J}_\omega(\vec{r}')$ is magnetic dipole moment

$\overleftrightarrow{Q}'_\omega = \int d^3r'\ 3\vec{r}'\vec{r}'\rho_\omega(\vec{r}')$ looks almost like electric quadrapole tensor to make it look like the proper quadrapole moment

$$\overleftrightarrow{Q}_\omega = \int d^3r' \left(3\vec{r}\,'\vec{r}\,' - r'^2 \overleftrightarrow{I}\right) \rho_\omega(\vec{r}\,')$$

we can write

$$\overleftrightarrow{Q}'_\omega = \overleftrightarrow{Q}_\omega + \overleftrightarrow{I} \int d^3r'\, r'^2 \rho_\omega(\vec{r}\,')$$

identity matrix $I_{ij} = \delta_{ij}$

$$\vec{I}_2 = -\hat{r} \times \vec{m}_\omega - \frac{1}{2} \frac{i\omega}{3c} \hat{r} \cdot \overleftrightarrow{Q}_\omega - \frac{i\omega}{6c} \hat{r}\, C(\omega)$$

where $C_\omega \equiv \int d^3r'\, r'^2 \rho_\omega(\vec{r}\,')$ is a scalar

Magnetic dipole approximation from $\vec{I}_2$

$$\boxed{\vec{A}_{M1}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \left(-\hat{r} \times \vec{m}_\omega\right)}$$

Electric quadrapole approximation from $\vec{I}_2$

$$\boxed{\vec{A}_{E2}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \left(-\frac{i\omega}{6c} \hat{r} \cdot \overleftrightarrow{Q}_\omega\right)}$$

The last piece $\frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right)\left(-\frac{i\omega}{6c} \hat{r}\, C_\omega\right)$

can always be ignored - it is a radial function and so its curl always vanishes $\Rightarrow$ gives no contribution to $\vec{B}$. Similarly, since $-\frac{i\omega}{c} \vec{E}_\omega = i\vec{k} \times \vec{B}_\omega$ by Ampere's law, this term will give no contribution to $\vec{E}$.

holds away from source where $\vec{J} \approx 0$.

So with these two approximations ① and ②

$$\vec{A}_\omega(\vec{r}) = \vec{A}_{E1}(\vec{r}) + \vec{A}_{M1}(\vec{r}) + \vec{A}_{E2}(\vec{r})$$

keeping higher order terms would give magnetic quadrupole, electric octopole etc.

Compare strengths of the terms above

Approx ③ ($r \gg \lambda$) Radiation zone: far from sources,

$\frac{1}{r} \ll k$ so $\left(\frac{1}{r} - ik\right) \approx -ik$ in $\vec{A}_{M1}$ and $\vec{A}_{E2}$

only keep terms of $O\left(\frac{1}{r}\right)$

electric dipole $\quad \vec{p}_\omega \sim qd \qquad \vec{A}_{E1} \sim qkd$

magnetic dipole $\quad \vec{m}_\omega \sim \frac{vqd}{c} \qquad \vec{A}_{M1} \sim qkd\left(\frac{v}{c}\right)$

use $v \sim \frac{d}{t} \sim d\omega \sim dkc \Rightarrow \vec{A}_{M1} \sim q(kd)^2$

electric quadrupole $\quad \overleftrightarrow{Q}_\omega \sim qd^2 \qquad \vec{A}_{E2} \sim qd^2k\frac{\omega}{c} \sim q(kd)^2$

Since Approx ② assumed $kd \approx \frac{v}{c} \ll 1$

above is expansion in powers of kd

leading term is electric dipole

next order are $\left[\begin{array}{l}\text{magnetic dipole} \\ \text{electric quadrupole}\end{array}\right]$ $\qquad \frac{A_{M1}}{A_{E1}} \sim \frac{A_{E2}}{A_{E1}} \sim kd$

next order terms are smaller than A_{E1} by factor $(kd)^2$

etc.

Electric dipole Approximation — the leading term in non-relativistic expansion

$$\vec{A}_{E1}(\vec{r}) = -ik\vec{p}_\omega \frac{e^{ikr}}{r} \qquad \vec{\nabla}\times(\varphi\vec{F}) = (\vec{\nabla}\varphi)\times\vec{F} + \varphi\vec{\nabla}\times\vec{F}$$

$$\vec{B}_{E1} = \vec{\nabla}\times\vec{A}_{E1} = -ik\left(\vec{\nabla}\frac{e^{ikr}}{r}\right)\times\vec{p}_\omega$$

$$= -ik\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega$$

$$= k^2\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega$$

In radiation zone approx, $kr \gg 1$

$$\boxed{\vec{B}_{E1} \simeq k^2\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega}$$

To get electric field, use Ampere's Law

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \text{(away from source where } \vec{J} = 0\text{)}$$

For oscillating fields $\vec{E} = E_\omega e^{-i\omega t}$

$$\vec{\nabla}\times\vec{B}_\omega = -\frac{i\omega}{c}\vec{E}_\omega \Rightarrow E_{E1} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{E1}$$

$$\vec{E}_{E1} = \frac{i}{k}\vec{\nabla}\times\left[\frac{k^2 e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega\right]$$

ignore in RZ approx

$$\vec{E}_{E1} = \frac{i}{k}\left(\vec{\nabla} e^{ikr}\right) \times \left[\frac{k^2}{r}\left(1+\frac{i}{kr}\right) \hat{r} \times \vec{p}_\omega \right]$$

$$+ \frac{i}{k} e^{ikr} \underbrace{\vec{\nabla} \times \left[\frac{k^2}{r}\left(1+\frac{i}{kr}\right) \hat{r} \times \vec{p}_\omega \right]}$$

this will always be of order $1/r^2$
so can ignore it in radiation zone approx

So in radiation zone approx

$$\vec{E}_{E1} = \left(\vec{\nabla} e^{ikr}\right) \times \left[\frac{ik}{r} \hat{r} \times \vec{p}_\omega \right]$$

$$\boxed{\vec{E}_{E1} = -k^2 \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{p}_\omega\right)}$$