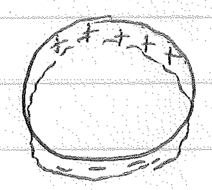
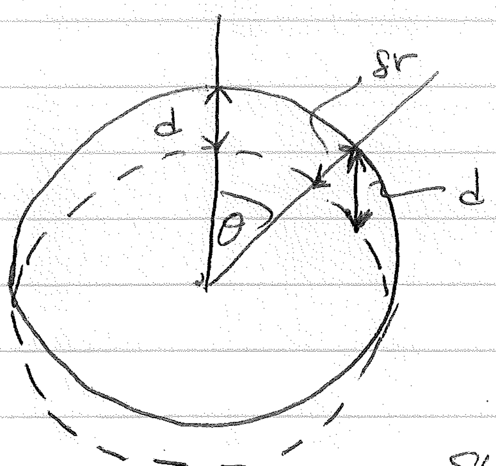
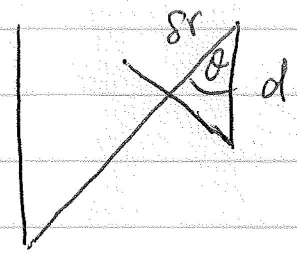
Physical example with $\sigma(\theta) = k\cos\theta$

Two spheres of radii R, with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement
This is a uniformly "polarized" sphere





$d\cos\theta = \delta r$

surface charge is $\sigma(\theta) = \rho\,\delta r$
$= \rho d \cos\theta$

$$\boxed{\sigma(\theta) = \rho d \cos\theta}$$

$k \equiv \rho d \equiv \mathbb{P}$

total dipole moment is $(\rho d)\frac{4}{3}\pi R^3$

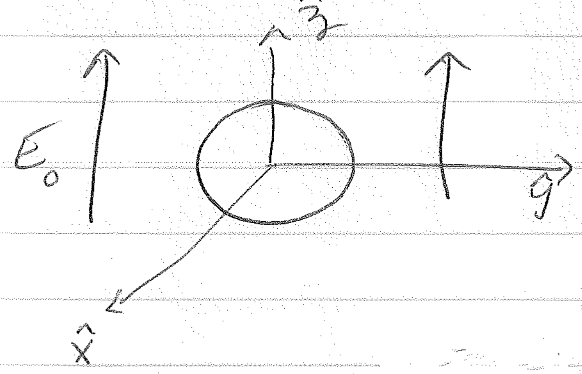
polarization $= \dfrac{\text{dipole moment}}{\text{volume}} = \rho d = \mathbb{P}$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}\hat{z} = -\frac{4\pi \mathbb{P}}{3}\hat{z} = -\frac{4\pi}{3}\vec{\mathbb{P}}$

Grounded

(3) Conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$
$= -E_0 r \cos\theta$



boundary conditions

$$\begin{cases} \phi(R,\theta) = 0 \\ \phi(r\to\infty,\theta) = -E_0 r\cos\theta \end{cases}$$

solution outside sphere has the form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

From boundary condition as $r \to \infty$ we have

$A_\ell = 0$ all $\ell \neq 1$

$A_1 = -E_0$ since $P_1(\cos\theta) = \cos\theta$

$$\phi(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

From $\phi(R,\theta) = 0$ we have

$$0 = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$\Rightarrow B_\ell = 0$ all $\ell \neq 1$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = E_0 R^3$$

So $$\boxed{\phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

1st tem is just potential $-E_0 r\cos\theta$ of the uniform applied electric field.

2nd tem is potential due to the induced surface charge on the surface - it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta$$

$$= 3E_0\cos\theta$$

$$\sigma(\theta) = \frac{3}{4\pi}E_0\cos\theta$$ like uniformly polarized sphere $k = \frac{3E_0}{4\pi} = P$

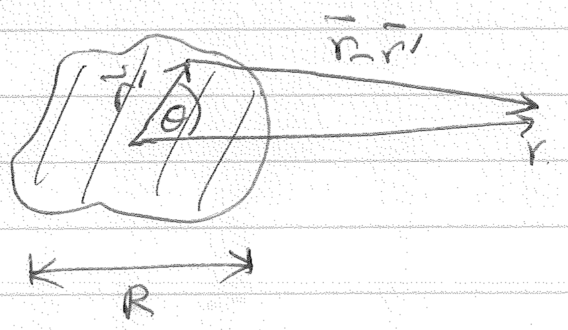
from ② we know that the field inside the sphere due to this σ is just $-\frac{4}{3}\pi k\hat{z} = -\frac{4}{3}\pi\frac{3E_0}{4\pi}\hat{z}$

$= -E_0\hat{z}$. This is just what is required so that the total field in the conducting sphere vanishes.

Can check that outside the sphere, $\vec{E} = -\vec{\nabla}\phi$ is normal to surface of sphere at $r=R$.

Multipole Expansion

region with $\rho \neq 0$

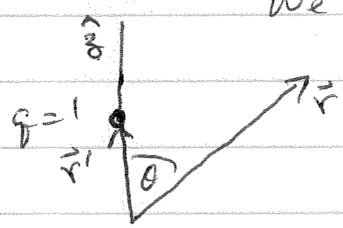


We want to find the potential ϕ for an arbitrary localized distribution of charge ρ, at distances far away $r >> R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \text{general Coulomb formula}$$

we want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r >> r'$

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ, so we can express it as an expansion in Legendre polynomials.

For $r > r'$,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad \text{all } A_\ell = 0 \text{ as need } \phi \to 0 \text{ as } r \to \infty$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} P_\ell(\cos\theta)$$

We know $\phi(r, \theta=0) = \dfrac{1}{r-r'}$ (for $r > r'$)

← scalers here since when $\theta = 0$, $\vec{r}$ and $\vec{r}\,'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r,0) = \frac{1}{r} \sum_\ell \frac{B_\ell}{r^\ell} P_\ell(1)$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} \quad \text{as } P_\ell(1) = 1$$

$$= \frac{1}{r} \frac{1}{(1 - r'/r)}$$ ← exact result from Coulomb

Now Taylor expansion $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$

$$\Rightarrow \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} = \frac{1}{r} \left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots \right)$$

$\Rightarrow \quad B_\ell = (r')^\ell$ is solution

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)}$$

So for the charge distribution ρ,

$$\phi(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \int d^3r' \, \frac{\rho(\vec{r}\,')}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \, \rho(\vec{r}\,') \, (r')^\ell \, P_\ell(\cos\theta)$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}\,'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r). It is *exact* provided one adds all the infinite l terms. In practice, one generally approximates by summing only up to some finite l.

__Note:__ in doing the integrals

$$\int d^3r' \, \rho(\vec{r}')(r')^l P_l(\cos\theta)$$

θ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat this integration every time we change $\vec{r}$.

We will find a way around this by
(i) first looking explicitly at the few lowest order terms
(ii) a general method involving spherical harmonics $Y_{lm}(\theta,\varphi)$

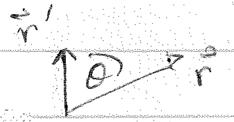
monopole: $l=0$ term

$$\phi^{(0)}(\vec{r}) = \frac{1}{r}\int d^3r'\,\rho(r') \qquad P_0(\cos\theta) = 1$$

$$= \frac{q}{r} \quad \text{where } q \equiv \int d^3r'\,\rho(r') \text{ is}$$
total charge

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}\,')\,r'\,P_1(\cos\theta)$$



$$= \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}\,')\,r'\cos\theta$$

Now $\vec{r}\cdot\vec{r}\,' = rr'\cos\theta \Rightarrow \hat{r}\cdot\vec{r}\,' = r'\cos\theta$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2}\,\hat{r}\cdot\int d^3r'\,\rho(\vec{r}\,')\,\vec{r}\,'$$

$$= \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \vec{p} \equiv \int d^3r'\,\rho(\vec{r}\,')\,\vec{r}\,'$$
is the dipole moment

For a set of point charges q_i at $\vec{r}_i$,
$$\vec{p} = \sum_i q_i \vec{r}_i$$

quadrapole: $\ell = 2$ term

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\,\rho(\vec{r}')\,r'^2\,P_2(\cos\theta)$$

$$= \frac{1}{r^3}\int d^3r'\,\rho(\vec{r}')\,r'^2\,\frac{1}{2}\left(3\cos^2\theta - 1\right)$$

use $\cos\theta = \hat{r}'\cdot\hat{r}$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\int d^3r'\,\rho(\vec{r}')\,\frac{1}{2}\left(3\,(\vec{r}'\cdot\hat{r})^2 - (r')^2\right)$$

$$= \frac{1}{r^3}\,\hat{r}\cdot\left[\int d^3r'\,\rho(\vec{r}')\,\frac{1}{2}\left(3\,\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)\right]\cdot\hat{r}$$

where $\overleftrightarrow{I}$ is the identity tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot\overleftrightarrow{I}\cdot\vec{v} = \vec{u}\cdot\vec{v}$
and $\vec{r}'\vec{r}'$ is the tensor such that for any two vectors $\vec{v}$ and $\vec{u}$, $\vec{u}\cdot[\vec{r}'\vec{r}']\cdot\vec{v} = (\vec{u}\cdot\vec{r}')(\vec{r}'\cdot\vec{v})$

Define quadrapole tensor $\overleftrightarrow{Q} \equiv \int d^3r'\,\rho(\vec{r}')\,\left(3\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{I}\right)$

$$\phi^{(2)}(\vec{r}) = \frac{1}{r^3}\,\frac{1}{2}\,\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}$$

So to lowest three terms

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} + \cdots$$

defined in terms of the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ of the charge distribution.

Note, the moments q, $\vec{p}$, $\overleftrightarrow{Q}$ do not depend on the observation point $\vec{r}$ — we can calculate them once and then use them to get $\phi(\vec{r})$ at all $\vec{r}$.

monopole: $q = \int d^3r\, \rho(\vec{r})$ scalar integral

dipole $\vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r}$ vector integral

$\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

if we pick a coordinate system, we have to do $\underline{\underline{3}}$ integrations to get the three component of $\vec{p}$

$$\hat{e}_i \cdot \vec{p} = p_i = \int d^3r\, \rho(\vec{r})\, r_i$$

quadrapole $\overleftrightarrow{Q} = \int d^3r\, \rho(\vec{r}) \left(3\vec{r}\,\vec{r} - (r')^2 \overleftrightarrow{I}\right)$ tensor integral

if we pick a coord system $x\ y\ z$ then $\overleftrightarrow{Q}$ is a matrix with components $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

$$\hat{e}_i \cdot \overleftrightarrow{Q} \cdot \hat{e}_j = Q_{ij} = \int d^3r\, \rho(\vec{r}) \left[3 r_i r_j - r^2 \delta_{ij}\right]$$

There are 9 elements of the 3×3 matrix Q_{ij}, but $Q_{ij} = Q_{ji}$ is symmetric so there are only 6 independent elements to compute.