Unit 3-2: Magnetic Materials: Magnetization Density M, Magnetic Field H, Bound Currents

Magnetic Materials

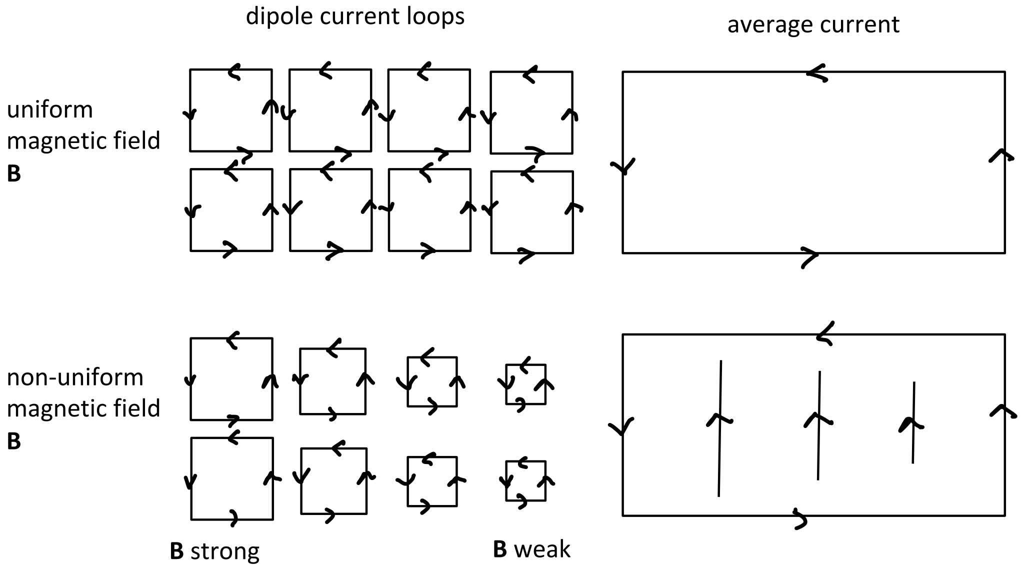
Circulating currents on the atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material. The sources of such circulating atomic currents are: (1) intrinsic angular momentum of electrons, i.e., "electron spin", and (2) orbital angular momentum of electrons. Adding such terms up, the atom can have a net total angular momentum, and so a net magnetic dipole moment, even if there is no magnetic field on the atom.

However, when $\mathbf{B} = 0$, these atomic magnetic dipole moments are generally in random orientations and so average to zero when averaging over many atomic length scales. In such a case, the application of a finite $\mathbf{B}$ causes the atomic magnetic dipole moments to tend to align, giving a net magnetization density $\mathbf{M}$ in the material. Unlike the electric polarization $\mathbf{P}$, which is always oriented in the same direction as $\mathbf{E}$, the magnetization $\mathbf{M}$ can be either aligned parallel or anti-parallel to the magnetic field. When $\mathbf{M} \propto \mathbf{B}$ the material is said to be *paramagnetic.* When $\mathbf{M} \propto -\mathbf{B}$ the material is said to be *diamagnetic.* Paramagnetism is usually associated with the intrinsic electron spin. Diamagnetism is usually associated with the electron orbital angular momentum, and can exist even if the total angular momentum of the electrons is zero, i.e., for atoms which have zero atomic dipole moment when $\mathbf{B} = 0$. The exception to this situation is the case of a ferromagnetic material in which the interaction between atomic magnetic moments causes them to align even when $\mathbf{B} = 0$. In this course we will not discuss more about the physical mechanisms leading to these three types of magnetic materials.

For the purpose of illustration, we will model atomic magnetic moments as small current loops with magnetic dipole moments $\mathbf{m}_i$. The magnetization density $\mathbf{M}$ is then defined as,

$$\mathbf{M}(\mathbf{r}, t) = \sum_i \mathbf{m}_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \tag{3.2.1}$$

We will consider the case of paramagnetic or diamagnetic materials. In this case, the application of a uniform applied $\mathbf{B}$ causes the $\mathbf{m}_i$ to tend to align, which gives rise to a current flowing around the boundary of the system. But if $\mathbf{B}$ is non-uniform, then in addition to the surface current one can create a finite current density $\mathbf{j}$ in the interior of the material. We can see these effects in the sketches shown below, where $\mathbf{B}$ is pointing perpendicular to the page.



When the applied $\mathbf{B}$ is uniform, then the interior currents from the aligned dipole current loops cancel, and only a uniform current flowing around the surface remains. When the applied $\mathbf{B}$ is non-uniform, varying from strong to weak as one moves to the right in the horizontal direction, then dipole current loops vary in size as one moves horizontally; the horizontal segments of the interior current loops again cancel, but the vertical segments do not, leading to an interior current density $\mathbf{j}$ in the vertical direction that decreases in magnitude as one moves to the right. The surface current remains, but it is now no longer uniform, decreasing in magnitude as one moves to the right.

We thus see that when there is a uniform magnetization $\mathbf{M}$ in the material, there is a surface current density flowing, but no interior volume current density. When $\mathbf{M}$ is non-uniform, however, in addition to a surface current there is also an interior volume current density that points in the direction orthogonal to both the direction of $\mathbf{M}$ and the direction in which $\mathbf{M}$ is varying.

Spatially Averaged Current Density, Magnetization Density, Magnetic Field H

To make the above picture more quantitative, we now carry out a spacial averaging of the microscopic current density $\mathbf{j}_0$, similar to what was done in the previous section for the microscopic charge density ρ_0.

$$\langle \mathbf{j}_0 \rangle = \left\langle \sum_{i\in\text{free}} q_i \mathbf{v}_i \delta(\mathbf{r}-\mathbf{r}_i) \right\rangle + \left\langle \sum_n \mathbf{j}_n \right\rangle \tag{3.2.2}$$

where the first sum above is from the externally added free charges in the material, and the second term is the current distribution associated with atom or molecule n of the material. For the current from atom/molecule n we can write,

$$\langle \mathbf{j}_n(\mathbf{r},t) \rangle = \left\langle \sum_{i\in n} q_i(\mathbf{v}_n + \mathbf{v}_{ni})\delta(\mathbf{r}-\mathbf{r}_n(t)-\mathbf{r}_{ni}(t)) \right\rangle = \sum_{i\in n} q_i(\mathbf{v}_n + \mathbf{v}_{ni}) f(\mathbf{r}-\mathbf{r}_n(t)-\mathbf{r}_{ni}(t)) \tag{3.2.3}$$

where the sum is over all the charges q_i in the atom/molecule, $\mathbf{v}_n = d\mathbf{r}_n/dt$ is the velocity of the atom/molecule's center of mass, and $\mathbf{v}_{ni} = d\mathbf{r}_{ni}/dt$ is the relative velocity of charge q_i with respect to the center of mass.

As we did for $\langle \rho_n \rangle$, we can expand the above expression in the $\mathbf{r}_{ni}$,

$$\langle \mathbf{j}_n \rangle = \sum_{i\in n} q_i(\mathbf{v}_n + \mathbf{v}_{ni}) \left[f(\mathbf{r}-\mathbf{r}_n) - \mathbf{r}_{ni}\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n) + \frac{1}{2}\sum_{\alpha,\beta} (\mathbf{r}_{ni})_\alpha (\mathbf{r}_{ni})_\beta \frac{\partial^2 f(\mathbf{r}-\mathbf{r}_n)}{\partial r_\alpha \partial r_\beta} + \dots \right] \tag{3.2.4}$$

We will keep only the first two terms in the above expansion. Multiplying out the factors, we get four terms in the above sum.

$$1) \qquad \sum_{i\in n} q_i \mathbf{v}_n f(\mathbf{r}-\mathbf{r}_n) \tag{3.2.5}$$

$$2) \qquad \sum_{i\in n} q_i \mathbf{v}_{ni} f(\mathbf{r}-\mathbf{r}_n) \tag{3.2.6}$$

$$3) \qquad -\sum_{i\in n} q_i \mathbf{v}_n \left[\mathbf{r}_{ni}\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right] \tag{3.2.7}$$

$$4) \qquad -\sum_{i\in n} q_i \mathbf{v}_{ni} \left[\mathbf{r}_{ni}\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right] \tag{3.2.8}$$

Consider first term (1).

$$1) \qquad \sum_{i\in n} q_i \mathbf{v}_n f(\mathbf{r}-\mathbf{r}_n) = \mathbf{v}_n f(\mathbf{r}-\mathbf{r}_n) \sum_{i\in n} q_i = q_n \mathbf{v}_n f(\mathbf{r}-\mathbf{r}_n) = \langle q_n \mathbf{v}_n \delta(\mathbf{r}-\mathbf{r}_n) \rangle \tag{3.2.9}$$

where $q_n = \sum_i q_i$ is just the total charge on atom/molecule n. This term (1) is just the current of the atom/molecule as if it were a point charge q_n. For a neutral atom/molecule, $q_n = 0$, and this term vanishes.

Now consider the second term (2). First note that, applying the chair rule with respect to the time derivative, we can write,

$$\frac{\partial}{\partial t}\langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n(t)) \rangle = \frac{\partial}{\partial t}\left[\sum_{i\in n} q_i \mathbf{r}_{ni}(t)\, f(\mathbf{r}-\mathbf{r}_n(t)) \right] = \sum_{i\in n} q_i \mathbf{v}_{ni} f(\mathbf{r}-\mathbf{r}_n) + \sum_{i\in n} q_i \mathbf{r}_{ni} \left[-\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\cdot\mathbf{v}_n\right] \tag{3.2.10}$$

where the last term in the above comes from $\partial f(\mathbf{r}-\mathbf{r}_n(t))/\partial t$. With this we can write for term (2)

$$2)\qquad \sum_{i\in n} q_i \mathbf{v}_{ni} f(\mathbf{r}-\mathbf{r}_n) = \frac{\partial}{\partial t}\left\langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n)\right\rangle + \left[\mathbf{v}_n\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right]\mathbf{p}_n \tag{3.2.11}$$

$$= \frac{\partial}{\partial t}\left\langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n)\right\rangle + (\mathbf{v}_n\cdot\boldsymbol{\nabla})\langle \mathbf{p}_n \delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.12}$$

where the second term means $\sum_\alpha v_{n,\alpha}\dfrac{\partial}{\partial r_\alpha}\langle \mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle$

Next, term (3) is,

$$3)\qquad -\sum_{i\in n} q_i \mathbf{v}_n \left[\mathbf{r}_{ni}\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right] = -\mathbf{v}_n\left[\mathbf{p}_n\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right] = -\mathbf{v}_n\boldsymbol{\nabla}\cdot\langle \mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.13}$$

$$= \sum_\alpha \mathbf{v}_n \frac{\partial}{\partial r_\alpha}\langle p_{n,\alpha}\delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.14}$$

and term (4) is,

$$4)\qquad -\sum_{i\in n} q_i \mathbf{v}_{ni}\left[\mathbf{r}_{ni}\cdot\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\right] = -\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\cdot\sum_{i\in n} q_i\,\mathbf{r}_{ni}\,\mathbf{v}_{ni} \tag{3.2.15}$$

Now we have seen the tensor $\sum_{i\in n} q_i \mathbf{r}_{ni}\mathbf{v}_{ni}$ before when we discussed the magnetic dipole approximation. With $\mathbf{j}(\mathbf{r}) = \sum_{i\in n} q_i \mathbf{v}_{ni}\delta(\mathbf{r}-\mathbf{r}_{ni})$, and using our earlier results from the magnetic dipole derivation, we have

$$\sum_{i\in n} q_i \mathbf{r}_{ni}\mathbf{v}_{ni} = \int d^3r\,\mathbf{r}\,\mathbf{j} = -\int d^3r\,\mathbf{j}\,\mathbf{r} - \int d^3r\,(\boldsymbol{\nabla}\cdot\mathbf{j})\,\mathbf{r}\,\mathbf{r} \tag{3.2.16}$$

When we did the magnetic dipole approximation we were discussing magnetostatics, where $\boldsymbol{\nabla}\cdot\mathbf{j}=0$, and so the last term above vanished. Here we are not limiting ourselves to statics, so the last term is not in general zero. But using charge conservation to write $\boldsymbol{\nabla}\cdot\mathbf{j} = -\partial\rho/\partial t$ we have,

$$\int d^3r\,\mathbf{r}\,\mathbf{j} = -\int d^3r\,\mathbf{j}\,\mathbf{r} + \int d^3r\,\frac{\partial\rho}{\partial t}\,\mathbf{r}\,\mathbf{r} = -\int d^3r\mathbf{j}\,\mathbf{r} + \frac{\partial}{\partial t}\left[\int d^3r\,\rho\,\mathbf{r}\,\mathbf{r}\right] \tag{3.2.17}$$

Although the last term in the above is not in general zero, it is a quadrupole term, $\sim (a_0/L)^2$, and so of the same order as the terms we dropped when we truncated our expansion in $\mathbf{r}_{ni}$ to *linear* order. We therefore ignore this last terms since it is negligibly small. And so

$$\int d^3r\,\mathbf{r}\,\mathbf{j} = -\int d^3\,r\mathbf{j}\,\mathbf{r} = \frac{1}{2}\int d^3r\,\left[\mathbf{r}\,\mathbf{j} - \mathbf{j}\,\mathbf{r}\right] \tag{3.2.18}$$

or, back in terms of the point charges,

$$\sum_{i\in n} q_i \mathbf{r}_{ni}\,\mathbf{v}_{ni} = \frac{1}{2}\sum_{i\in n} q_i\left[\mathbf{r}_{ni}\,\mathbf{v}_{ni} - \mathbf{v}_{ni}\,\mathbf{r}_{ni}\right] \tag{3.2.19}$$

So now we can write for term (4)

$$(4) \qquad -\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\cdot\sum_{i\in n} q_i\,\mathbf{r}_{ni}\,\mathbf{v}_{ni} = -\boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\cdot\frac{1}{2}\sum_{i\in n} q_i\left[\mathbf{r}_{ni}\,\mathbf{v}_{ni}-\mathbf{v}_{ni}\,\mathbf{r}_{ni}\right] \tag{3.2.20}$$

$$= \frac{1}{2}\sum_{i\in n} q_i\left[(\boldsymbol{\nabla} f\cdot\mathbf{r}_{ni})\,\mathbf{v}_{ni}-(\boldsymbol{\nabla} f\cdot\mathbf{v}_{ni})\,\mathbf{r}_{ni}\right] \tag{3.2.21}$$

$$-\frac{1}{2}\sum_{i\in n} q_i\,\boldsymbol{\nabla} f\times(\mathbf{v}_{ni}\times\mathbf{r}_{ni}) \quad \text{using the triple product rule} \tag{3.2.22}$$

$$= \boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\times\frac{1}{2}\sum_{i\in n}\mathbf{r}_{ni}\times q_i\mathbf{v}_{ni} \tag{3.2.23}$$

$$= \boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\times\frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{j} \tag{3.2.24}$$

$$= \boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\times c\,\mathbf{m}_n \tag{3.2.25}$$

where $\mathbf{m}_n = \frac{1}{2c}\sum_{i\in n}\mathbf{r}_{ni}\times q_i\mathbf{v}_{ni}$ is the magnetic dipole moment $\mathbf{m}_n$ of atom/molecule n. So finally,

$$(4) \quad = \boldsymbol{\nabla} f(\mathbf{r}-\mathbf{r}_n)\times c\,\mathbf{m}_n = \boldsymbol{\nabla}\times\left[f(\mathbf{r}-\mathbf{r}_n)\,c\,\mathbf{m}_n\right] = \boldsymbol{\nabla}\times\langle c\,\mathbf{m}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.26}$$

Adding all the pieces (1) + (2) + (3) + (4) we get

$$\langle\mathbf{j}_n\rangle = \langle q_n\mathbf{v}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle + c\,\boldsymbol{\nabla}\times\langle\mathbf{m}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle + \frac{\partial}{\partial t}\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle + (\mathbf{v}_n\cdot\boldsymbol{\nabla})\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle - \mathbf{v}_n\boldsymbol{\nabla}\cdot\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.27}$$

where the five terms above come from (1), (4), (2), (2), and (3) respectively.

Now defining the average magnetization density,

$$\mathbf{M}(\mathbf{r},t) = \sum_n\langle\mathbf{m}_n\delta(\mathbf{r}-\mathbf{r}_n(t))\rangle \tag{3.2.28}$$

and using the polarization density as before,

$$\mathbf{P}(\mathbf{r},t) = \sum_n\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}(t))\rangle \tag{3.2.29}$$

we have for the total bound current from the atom/molecules that comprise the material,

$$\sum_n\langle\mathbf{j}_n\rangle = \sum_n\langle q_n\mathbf{v}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle + c\boldsymbol{\nabla}\times\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t} \tag{3.2.30}$$

$$+\sum_n\Big[(\mathbf{v}_n\cdot\boldsymbol{\nabla})\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle - \mathbf{v}_n\boldsymbol{\nabla}\cdot\langle\mathbf{p}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle\Big] \tag{3.2.31}$$

The last sum in Eq. (3.2.31) is usually small and can be ignored. This is because the molecular velocities $\mathbf{v}_n$ are usually small and randomly oriented, so that they average to zero.

We thus define the macroscopic current density,

$$\mathbf{j}(\mathbf{r},t) = \sum_{i\in\text{free}}\langle q_i\mathbf{v}_i\delta(\mathbf{r}-\mathbf{r}_i)\rangle + \sum_n\langle q_n\mathbf{v}_n\delta(\mathbf{r}-\mathbf{r}_n)\rangle \tag{3.2.32}$$

where the first term is the current from the external free charges that are added to the material, and the second term is the current associated with the motion of the atoms/molecules that comprise the material. If those particles are charge neutral with $q_n = 0$, then the second term vanishes and the macroscopic current density is the same as the free current density.

Then we have for the total average microscopic current,

$$\langle \mathbf{j}_0 \rangle = \mathbf{j} + c\boldsymbol{\nabla}\times\mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \tag{3.2.33}$$

The term $c\boldsymbol{\nabla}\times\mathbf{M}$ might have been anticipated from our discussion and sketch at the beginning of this section. The term $\partial\mathbf{P}/\partial t$ we will show is crucial for charge conservation.

The Macroscopic Ampere's Law with Maxwell's Correction then becomes,

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\langle \mathbf{j}_0 \rangle + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} + 4\pi\boldsymbol{\nabla}\times\mathbf{M} + \frac{4\pi}{c}\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{3.2.34}$$

Rearranging terms we get,

$$\boldsymbol{\nabla}\times(\mathbf{B} - 4\pi\mathbf{M}) = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E} + 4\pi\mathbf{P}) \tag{3.2.35}$$

Defining,

$$\boxed{\mathbf{H} \equiv \mathbf{B} - 4\pi\mathbf{M}}, \qquad \text{and using our earlier definition} \quad \mathbf{D} \equiv \mathbf{E} + 4\pi\mathbf{P} \tag{3.2.36}$$

we get the Macroscopic Ampere's Law,

$$\boxed{\boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}} \tag{3.2.37}$$

The official terminology is that **B** is the magnetic *induction*, while **H** is the magnetic *field*. However, following common usage, we will refer to both **B** and **H** as the magnetic field, and you will know which one is meant by the symbol that is used.

Conservation of Bound Charge and Conservation of Macroscopic Charge

The above derivation was based on a classical picture of the motion of charges within the atoms/molecules comprising the material. When these atoms/molecules have an intrinsic magnetic moment due to electron spin, we can add these to **M** in an obvious way.

When the material is comprised of atoms/molecules that are neutral, with $q_n = 0$, the *bound current* is given by,

$$\mathbf{j}_b = \sum_n \langle \mathbf{j}_n \rangle = c\boldsymbol{\nabla}\times\mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \tag{3.2.38}$$

The term $\partial\mathbf{P}/\partial t$ is crucial for conservation of the bound charge, as we see as follows,

$$\boldsymbol{\nabla}\cdot\mathbf{j}_b = c\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{M}) + \boldsymbol{\nabla}\cdot\frac{\partial \mathbf{P}}{\partial t} = 0 + \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{\partial \rho_b}{\partial t} \tag{3.2.39}$$

so,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{j}_b + \frac{\partial \rho_b}{\partial t} = 0} \tag{3.2.40}$$

Since the total average charge must be conserved,

$$\boldsymbol{\nabla}\cdot\langle \mathbf{j}_0 \rangle + \frac{\partial \langle \rho_0 \rangle}{\partial t} = 0 \tag{3.2.41}$$

and since $\langle\mathbf{j}_0\rangle$ and $\langle\rho_0\rangle$ are the sum of the corresponding macroscopic and bound terms, $\langle\mathbf{j}_0\rangle = \mathbf{j}+\mathbf{j}_b$ and $\langle\rho_0\rangle = \rho+\rho_b$, then it also follows that the macroscopic charge is conserved,

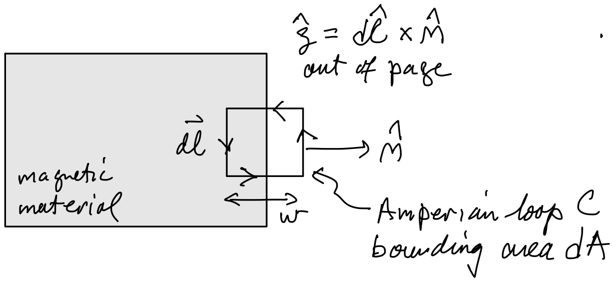
$$\boldsymbol{\nabla}\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0 \tag{3.2.42}$$

Bound Currents

For a *magnetostatic* situation, a spatially varying magnetization density gives rise to an internal bound volume current density $\mathbf{j}_b$, just as we saw in the sketch at the beginning of this section.

$$\boxed{\mathbf{j}_b = c\boldsymbol{\nabla}\times\mathbf{M}} \tag{3.2.43}$$

Now we see how the magnetization density also leads to bound surface currents.



Consider a small Amperian loop C piercing the surface of the material, as shown in the sketch above. The loop C bounds a surface $\mathcal{S}$ of area dA whose normal is in the direction $\hat{\mathbf{z}} = d\hat{\boldsymbol{\ell}}\times\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward pointing normal to the surface and $d\hat{\boldsymbol{\ell}}$ lies in the plane of the surface, as shown in the sketch. Integrate over the surface $\mathcal{S}$, taking the width $w\to 0$. One then has,

$$c\int_{\mathcal{S}} da\,\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}\times\mathbf{M}) = \int_{\mathcal{S}}\hat{\mathbf{z}}\cdot\left[\mathbf{j}_b - \frac{\partial\mathbf{P}}{\partial t}\right] = dA\,\hat{\mathbf{z}}\cdot\left[\mathbf{j}_b - \frac{\partial\mathbf{P}}{\partial t}\right] = dA\,\hat{\mathbf{z}}\cdot\mathbf{j}_b = (w\,d\boldsymbol{\ell}\times\hat{\mathbf{n}})\cdot\mathbf{j}_b = (\hat{\mathbf{n}}\times w\mathbf{j}_b)\cdot d\boldsymbol{\ell} \tag{3.2.44}$$

In the above we used that the $\partial\mathbf{P}/\partial t$ term vanishes since the area dA vanishes as $w\to 0$, while $\mathbf{P}$ stays finite. We also used that $dA = wd\ell$ and $\hat{\mathbf{z}} = d\hat{\boldsymbol{\ell}}\times\hat{\mathbf{n}}$, so that $dA\,\hat{\mathbf{z}} = wd\boldsymbol{\ell}\times\hat{\mathbf{n}}$. Finally, we used the general relation that $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$.

In the above, the final term involving $\mathbf{j}_b$ would vanish as $w\to 0$, unless $\mathbf{j}_b$ is infinite at the surface in just the right way that $w\mathbf{j}_b$ stays finite. And that is just what happens when there is a bound surface current $\mathbf{K}_b = w\mathbf{j}_b$. To see this, we use Stoke's Theorem to write,

$$c\int_{\mathcal{S}} da\,\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}\times\mathbf{M}) = c\oint_C d\boldsymbol{\ell}\cdot\mathbf{M} = c\,d\boldsymbol{\ell}\cdot\mathbf{M} \qquad \text{which is finite for finite } d\ell \tag{3.2.45}$$

The last step follows since, as $w\to 0$, it is only the inside side of the loop, parallel to the surface, that gives a non-zero contribution.

Equating the last two equations then gives,

$$(\hat{\mathbf{n}}\times w\mathbf{j}_b)\cdot d\boldsymbol{\ell} = (\hat{\mathbf{n}}\times\mathbf{K}_b)\cdot d\boldsymbol{\ell} = c\mathbf{M}\cdot d\boldsymbol{\ell} \qquad \text{with} \qquad \mathbf{K}_b = w\mathbf{j}_b \quad \text{the bound surface current} \tag{3.2.46}$$

The above holds for any $d\boldsymbol{\ell}$ in the plane of the surface, so we conclude,

$$c\mathbf{M}_t = \hat{\mathbf{n}}\times K_b \qquad \text{where } \mathbf{M}_t \text{ is the component of } \mathbf{M} \text{ tangent to the plane of the material's surface} \tag{3.2.47}$$

Finally, we can the the cross product of both sides of the above with $\hat{\mathbf{n}}$ to get

$$c\hat{\mathbf{n}}\times\mathbf{M}_t = c\hat{\mathbf{n}}\times\mathbf{M} = \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{K}_b) = -\mathbf{K}_b \tag{3.2.48}$$

where the last step holds because $\hat{\mathbf{n}}$ is perpendicular to the surface, and hence perpendicular to $\mathbf{K}_b$.

To conclude, we have for the bound currents,

$$\boxed{\mathbf{j}_b = c\boldsymbol{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}, \qquad \mathbf{K}_b = c\mathbf{M} \times \hat{\mathbf{n}}} \tag{3.2.49}$$

Discussion Question 3.2

For a material in which the constituent particles are charge neutral, show that the total bound charge, integrated over the entire material, vanishes. Similarly, for a static situation, show that the total bound current flowing through a cross-sectional area of the material vanishes. Why is this what you should expect?