Unit 3-3: The Macroscopic Maxwell Equations

From the preceding discussions, we can now write the complete set of Macroscopic Maxwell Equations,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \tag{3.3.1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \tag{3.3.2}$$

where ρ and $\mathbf{j}$ are the *macroscopic* charge and current densities that do *not* include the bound charges or currents.

$\mathbf{D}$ and $\mathbf{H}$ are related to $\mathbf{E}$ and $\mathbf{B}$ by,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \text{with } \mathbf{P} \text{ the polarization density} \tag{3.3.3}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}, \qquad \text{with } \mathbf{M} \text{ the magnetization density} \tag{3.3.4}$$

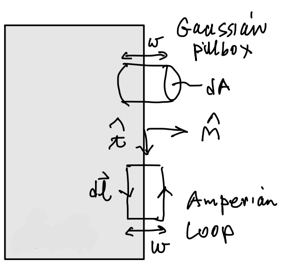
A good way to remember which equations have the $\mathbf{E}$ and $\mathbf{B}$, and which have the $\mathbf{D}$ and $\mathbf{H}$, is to note that it is the *homogeneous* equations, which do not involve the charge and current sources, that involve $\mathbf{E}$ and $\mathbf{B}$. It is the *inhomogeneous* equations, which do involve the sources, that involve $\mathbf{D}$ and $\mathbf{H}$, because these are the equations that include the effects of the bound charges and currents via $\mathbf{P}$ and $\mathbf{M}$.

Boundary Conditions

We now want to discuss how the macroscopic fields behave at charged surfaces and at current sheets. The arguments are similar to what we saw before for the microscopic fields.

Electric field $\mathbf{E}$ and electric displacement $\mathbf{D}$

At the surface of a dielectric, or at the interface between two different dielectrics:



We can integrate the flux of $\mathbf{D}$ over the surface $\mathcal{S}$ of a small Gaussian pillbox of area dA and width w. Taking $w \to 0$, and using $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho$, we get

$$\oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{D} = dA\,\hat{\mathbf{n}}\cdot(\mathbf{D}^{\text{above}} - \mathbf{D}^{\text{below}}) = 4\pi Q_{\text{encl}} = 4\pi\sigma\,dA \tag{3.3.5}$$

where $\hat{\mathbf{n}}$ is the outward normal to the dielectric surface. Note, since $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho$ with ρ the *macroscopic* charge density (does not include the bound charges), then Q_{encl} above is similarly the total macroscopic charge within the pillbox, and σ is the *macroscopic* surface charge on the surface (does not include any bound σ_b)

Integrating around the Amperian loop C bounding the surface $\mathcal{S}$, and using $\boldsymbol{\nabla}\times\mathbf{E} = -(1/c)(\partial B/\partial t)$, we get,

$$\oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = \int_{\mathcal{S}} da\,\hat{\mathbf{n}}_s\cdot(\boldsymbol{\nabla}\times\mathbf{E}) = -\frac{1}{c}\frac{d}{dt}\int_{\mathcal{S}} da\,\hat{\mathbf{n}}_s\cdot\mathbf{B} = 0 \quad \text{as } w\to 0 \tag{3.3.6}$$

In the above, $\hat{\mathbf{n}}_s$ is the normal to the surface bounded by the loop. We then have, as $w \to 0$,

$$0 = \oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = \boldsymbol{d\ell}\cdot(\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}}) \tag{3.3.7}$$

From the above we conclude that the *normal* component of $\mathbf{D}$ jumps by $4\pi\sigma$, while the *tangential* component of $\mathbf{E}$ is continuous at the surface. If there is no free charge on the surface (so $\sigma = 0$, though there might still be σ_b on the surface), then the normal component of $\mathbf{D}$ is continuous.

Magnetic fields $\mathbf{B}$ and $\mathbf{H}$

At the surface of a magnetic material, or at the interface between two different magnetic materials:

Integrating over the Gaussian pillbox, taking the width $w \to 0$, and using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, we get

$$\oint_{\mathcal{S}} da\, \hat{\mathbf{n}} \cdot \mathbf{B} = dA\, \hat{\mathbf{n}} \cdot (\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}}) = 0 \tag{3.3.8}$$

Integrating over the Amperian loop, taking $w \to 0$, and using $\boldsymbol{\nabla} \times \mathbf{H} = (4\pi/c)\mathbf{j} + (1/c)(\partial \mathbf{D}/\partial t)$, we get

$$\oint_{C} \boldsymbol{d\ell} \cdot \mathbf{H} = \int_{\mathcal{S}} da\, \hat{\mathbf{n}}_s \cdot (\boldsymbol{\nabla} \times \mathbf{H}) = \int_{\mathcal{S}} da\, \hat{\mathbf{n}}_s \cdot \left[\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right] = \frac{4\pi}{c} d\ell\, w\, \hat{\mathbf{n}}_s \cdot \mathbf{j} = \frac{4\pi}{c} d\ell\, \hat{\mathbf{n}}_s \cdot \mathbf{K} \tag{3.3.9}$$

where $w\mathbf{j} = \mathbf{K}$ is the macroscopic sheet current flowing on the surface. The term involving $\mathbf{D}$ vanishes since $\mathbf{D}$ stays finite but the area vanishes as $w \to 0$. The term involving $\mathbf{j}$ will also vanish if there is no sheet current $\mathbf{K}$, but will not vanish if there is. Next, as $w \to 0$,

$$\oint_{C} \boldsymbol{d\ell} \cdot \mathbf{H} = \boldsymbol{d\ell} \cdot (\mathbf{H}^{\text{below}} - \mathbf{H}^{\text{above}}) = \frac{4\pi}{c} d\ell\, \hat{\mathbf{n}}_s \cdot \mathbf{K} = \frac{4\pi}{c}(\boldsymbol{d\ell} \times \hat{\mathbf{n}}) \cdot \mathbf{K} = \frac{4\pi}{c}\boldsymbol{d\ell} \cdot (\hat{\mathbf{n}} \times \mathbf{K}) \tag{3.3.10}$$

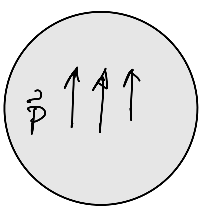
where we used $\hat{\mathbf{n}}_s = \boldsymbol{d\hat{\ell}} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to the material's surface, as in the sketch. Finally, since $\boldsymbol{d\ell}$ can point in any direction tangent to the material's surface, we can write,

$$(\mathbf{H}^{\text{above}} - \mathbf{H}^{\text{below}})_{\text{tangent}} = \frac{4\pi}{c}(\mathbf{K} \times \hat{\mathbf{n}}) \tag{3.3.11}$$

We conclude that the *normal* component of $\mathbf{B}$ is continuous at the surface, while the *tangential* component of $\mathbf{H}$ jumps by $(4\pi/c)(\mathbf{K} \times \hat{\mathbf{n}})$. If there is no free sheet current at the surface (so $\mathbf{K} = 0$, though there might still be a $\mathbf{K}_b$ on the surface), then the tangential component of $\mathbf{H}$ is continuous.

Examples

1) Uniformly polarized sphere of radius R $\qquad \mathbf{P} = P\hat{\mathbf{z}}$



The bound volume charge density is $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0$, since $\mathbf{P}$ is constant. The bound surface charge density is $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot P\hat{\mathbf{z}} = P\cos\theta$.

We saw earlier in Notes 2-3 that a sphere with a surface charge $\sigma = \sigma_0 \cos\theta$ gives an electric field that looks like a pure dipole field for $r > R$ outside, and is a constant for $r < R$ inside. Using that result with $\sigma_0 = P$ we get

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{4\pi}{3} R^3 P \left[\dfrac{2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}}{r^3}\right] & r > R \\ -\dfrac{4\pi}{3} P\, \hat{\mathbf{z}} & r < R \end{cases} \tag{3.3.12}$$

The total dipole moment on the sphere is $\mathbf{p} = (4\pi/3)R^3\mathbf{P}$.

Let us now check that this $\mathbf{E}$ satisfies the above boundary conditions.

The tangential component of $\mathbf{E}$ is,

$$\mathbf{E}_t^{\text{above}} = \hat{\boldsymbol{\theta}} \cdot \mathbf{E}^{\text{above}} = \left(\frac{4\pi}{3}R^3 P\right)\frac{\sin\theta}{R^3}\,\hat{\boldsymbol{\theta}} = \frac{4\pi}{3}P\sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.13}$$

$$\mathbf{E}_t^{\text{below}} = \hat{\boldsymbol{\theta}} \cdot \mathbf{E}^{\text{below}} = -\frac{4\pi}{3}P(\hat{\mathbf{z}} \cdot \hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = \frac{4\pi}{3}P\sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.14}$$

So the tangential component of $\mathbf{E}$ is indeed continuous like it should be.

Now we check the normal component of $\mathbf{D}$.

Outside, $\mathbf{P} = 0$, so $\mathbf{D} = \mathbf{E}$.

Inside, $\mathbf{E} = -(4\pi/3)\mathbf{P} \quad \Rightarrow \quad \mathbf{P} = -(3/4\pi)\mathbf{E} \quad \Rightarrow \quad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \mathbf{E} - 3\mathbf{E} = -2\mathbf{E} = (8\pi/3)P\hat{\mathbf{z}}$.

So the normal component of $\mathbf{D}$ is,

$$\hat{\mathbf{r}} \cdot \mathbf{D}^{\text{above}} = \hat{\mathbf{r}} \cdot \mathbf{E}^{\text{above}} = \frac{4\pi}{3} R^3 P \frac{2\cos\theta}{R^3} = \frac{8\pi}{3} P\cos\theta \tag{3.3.15}$$

$$\hat{\mathbf{r}} \cdot \mathbf{D}^{\text{below}} = \hat{\mathbf{r}} \cdot \frac{8\pi}{3} P\hat{\mathbf{z}} = \frac{8\pi}{3} P\cos\theta \tag{3.3.16}$$

Since there is no free charge on the surface, $\sigma = 0$, and the normal component of $\mathbf{D}$ should be continuous, as we see above that it is.

Note, although the normal component of $\mathbf{D}$ is continuous, the normal component of $\mathbf{E}$ should have a discontinuity determined by the value of the *total* surface charge on the surface of the sphere. Since there is no free charge on the surface, the total charge on the surface is just equal to the bound surface charge, $\sigma_b = \hat{\mathbf{r}} \cdot \mathbf{P} = P\,\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = P\cos\theta$. To check this,

$$\hat{\mathbf{r}} \cdot \mathbf{E}^{\text{above}} = \frac{8\pi}{3} P\cos\theta \qquad \text{as before} \tag{3.3.17}$$

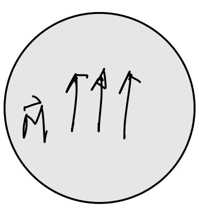
$$\hat{\mathbf{r}} \cdot \mathbf{E}^{\text{below}} = \hat{\mathbf{r}} \cdot \left(-\frac{4\pi}{3} P\hat{\mathbf{z}}\right) = -\frac{4\pi}{3} P\cos\theta \tag{3.3.18}$$

so

$$\hat{\mathbf{r}} \cdot (\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}}) = \frac{8\pi}{3} P\cos\theta + \frac{4\pi}{3} P\cos\theta = 4\pi P\cos\theta = 4\pi\sigma_b \tag{3.3.19}$$

as it should be.

2) Uniformly magnetized sphere of radius R $\qquad \mathbf{M} = M\hat{\mathbf{z}}$



The bound volume current density is $\mathbf{j}_b = c\boldsymbol{\nabla} \times \mathbf{M} = 0$, since $\mathbf{M}$ is a constant. The bound surface current is $\mathbf{K}_b = c\mathbf{M} \times \hat{\mathbf{n}} = cM(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = cM\sin\theta\,\hat{\boldsymbol{\varphi}}$.

In one of the Problem Sets you solved for the magnetic field $\mathbf{B}$ from a sphere with a surface current $\mathbf{K} = K_0 \sin\theta\,\hat{\boldsymbol{\varphi}}$. Outside the field looks like that of a pure magnetic dipole, while inside the field is a constant. Using that result with $K_0 = cM$ we have,

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{4\pi}{3} R^3 M \left[\dfrac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}\right] & r > R \\[2ex] \dfrac{8\pi}{3} M\,\hat{\mathbf{z}} & r < R \end{cases} \tag{3.3.20}$$

The total magnetic dipole moment on the sphere is $\mathbf{m} = (4\pi/3)\mathbf{M}$.

Let us now check that this $\mathbf{B}$ satisfies the appropriate boundary conditions.

The normal component of $\mathbf{B}$ is,

$$\hat{\mathbf{n}} \cdot \mathbf{B}^{\text{above}} = \hat{\mathbf{r}} \cdot \mathbf{B}^{\text{above}} = \frac{8\pi}{3} M\cos\theta \tag{3.3.21}$$

$$\hat{\mathbf{n}} \cdot \mathbf{B}^{\text{below}} = \hat{\mathbf{r}} \cdot \mathbf{B}^{\text{below}} = \frac{8\pi}{3} M\,(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = \frac{8\pi}{3} M\cos\theta \tag{3.3.22}$$

So the normal component of $\mathbf{B}$ is indeed continuous as it should be.

Now we check the tangential component of $\mathbf{H}$.

Outside, $\mathbf{M} = 0$, so $\mathbf{H} = \mathbf{B}$.

Inside, $\mathbf{B} = (8\pi/3)\mathbf{M} \quad \Rightarrow \quad \mathbf{M} = (3/8\pi)\mathbf{B} \quad \Rightarrow \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \mathbf{B} - (3/2)\mathbf{B} = -(1/2)\mathbf{B} = -(4\pi/3)M\hat{\mathbf{z}}$.

So the tangential component of $\mathbf{H}$ is,

$$\mathbf{H}_t^{\text{above}} = \mathbf{B}_t^{\text{above}} = (\mathbf{B}^{\text{above}} \cdot \hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = \frac{4\pi}{3} M \sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.23}$$

$$\mathbf{H}_t^{\text{below}} = -\frac{4\pi}{3} M(\hat{\mathbf{z}} \cdot \hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = \frac{4\pi}{3} M \sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.24}$$

Since there is no free current on the surface, $\mathbf{K} = 0$, and the tangential component of $\mathbf{H}$ should be continuous, as we see above that it is.

Although the tangential component of $\mathbf{H}$ is continuous, the tangential component of $\mathbf{B}$ should have a discontinuity determined by the value of the *total* surface current on the surface of the sphere. Since there is no free current on the surface, the total current on the surface is just equal to the bound surface current, $\mathbf{K}_b = c\mathbf{M} \times \hat{\mathbf{n}} = cM(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = cM \sin\theta\,\hat{\boldsymbol{\varphi}}$. To check this,

$$\mathbf{B}_t^{\text{above}} = \frac{4\pi}{3} M \sin\theta\,\hat{\boldsymbol{\theta}} \qquad \text{as before} \tag{3.3.25}$$

$$\mathbf{B}_t^{\text{below}} = \frac{8\pi}{3} M(\hat{\mathbf{z}} \cdot \hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = -\frac{8\pi}{3} M \sin\theta\hat{\boldsymbol{\theta}} \tag{3.3.26}$$

so

$$\mathbf{B}_t^{\text{above}} - \mathbf{B}_t^{\text{below}} = \frac{4\pi}{3} M \sin\theta\,\hat{\boldsymbol{\theta}} + \frac{8\pi}{3} M \sin\theta\,\hat{\boldsymbol{\theta}} = 4\pi M \sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.27}$$

while

$$\frac{4\pi}{c}(\mathbf{K}_b \times \hat{\mathbf{n}}) = \frac{4\pi}{c} cM \sin\theta\,(\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}}) = 4\pi M \sin\theta\,\hat{\boldsymbol{\theta}} \tag{3.3.28}$$

So we have,

$$\mathbf{B}_t^{\text{above}} - \mathbf{B}_t^{\text{below}} = \frac{4\pi}{c}(\mathbf{K}_b \times \hat{\mathbf{n}}) \tag{3.3.29}$$

as it should be.