## Unit 5: Electromagnetic Waves in Dielectrics and Conductors, Polarization, Interfaces

In this unit we consider the propagation of electromagnetic waves in matter, using the macroscopic Maxwell's equations. We will consider waves in dielectrics and waves in conductors, considering the response as frequency varies. We also discuss the different types of polarization of a wave, i.e., linear, circular, and elliptical. We then discuss reflection and transmission when a wave hits an interface between two different types of media

## Unit 5-1: Electromagnetic Plane Waves in Dielectrics

Here we will focus on the propagation of electromagnetic waves in dielectric materials. In these notes, we will be assuming for simplicity that $\mu$ is a constant. We will soon see, however, that $\epsilon$ is not.

### The simple view that is *not* generally correct

The macroscopic Maxwell's equations with no sources, i.e., macroscopic charge $\rho = 0$ and current $\mathbf{j} = 0$, are,

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\mathbf{D} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \\ &\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{aligned} \tag{5.1.1}$$

We will assume we are dealing with linear materials,

$$\mathbf{B} = \mu\mathbf{H} \qquad \mathbf{D} = \epsilon\mathbf{E} \tag{5.1.2}$$

If $\mu$ and $\epsilon$ were simply constants, then the above Maxwell's equations would become

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t} \\ &\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{aligned} \tag{5.1.3}$$

Then,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = -\nabla^2\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{\mu\epsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{5.1.4}$$

which is just the wave equation for $\mathbf{E}$ with wave speed $v = \dfrac{c}{\sqrt{\mu\epsilon}} < c$.

Repeating all the calculations we did in Notes 4-4 for waves in a vacuum, we would find for harmonic plane waves,

$$\begin{aligned} &\omega^2 = \frac{c^2k^2}{\mu\epsilon} && \text{dispersion relation} \\ &\mathbf{E}_\mathbf{k} \perp \mathbf{k} && \mathbf{E} \text{ is perpendicular to the direction of wave propagation} \\ &\mathbf{B}_\mathbf{k} \perp \mathbf{k} && \mathbf{B} \text{ is perpendicular to the direction of wave propagation} \\ &\mathbf{B}_\mathbf{k} = \sqrt{\mu\epsilon}\,\hat{\mathbf{k}}\times\mathbf{E}_\mathbf{k} && \mathbf{B}\perp\mathbf{E}\text{, but now } |\mathbf{B}_\mathbf{k}| > |\mathbf{E}_\mathbf{k}| \\ &v = \frac{c}{\sqrt{\mu\epsilon}} < c && \text{speed of wave slows down compared to vacuum} \end{aligned} \tag{5.1.5}$$

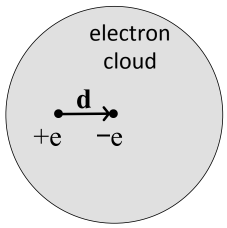
The behavior of electromagnetic waves in matter would be pretty much like electromagnetic waves in a vacuum, except for the small changes above due to the factor $\mu\epsilon$.

We will soon see, however, that this is not true. Even for linear materials, EM waves in matter show much richer phenomena than waves in the vacuum, and this is because $\mu$ and $\epsilon$ (and we will be focusing in particular on $\epsilon$) should not be taken as simple constants.

**Discussion Question 5.1**

How do you know from your every-day experience (so no math, no equations, no fancy technical words!) that the above cannot be true – that the behavior of EM waves in matter must be more complicated than the behavior of waves in the vacuum?

**Time Dependent Atomic Polarizability**




Consider as a simple model of an atom a positive point ion of charge $+e$ surrounded by a uniform spherical cloud of electronic charge of net charge $-e$ and radius $a_0$. The electric field inside the spherical electron cloud due to the electron is just that of a uniformly charged sphere. Taking the origin as the center of the spherical electron cloud, the electric field *inside* the cloud is,

$$\mathbf{E}(\mathbf{r}) = -\frac{e\mathbf{r}}{a_0^3} \qquad \text{note, } \mathbf{E} \text{ increases linearly in } \mathbf{r} \text{ as one moves away from the origin.} \tag{5.1.6}$$

So if the ion is displace a distance $-\mathbf{d}$ from the origin, it feels a restoring force pulling it back, $\mathbf{F} = +e\mathbf{E}(-\mathbf{d}) = \dfrac{e^2\,\mathbf{d}}{a_0^3}$. The restoring force on the electron is equal and opposite, so the force on the electron cloud, when it displaced $\mathbf{d}$ from the ion, is,

$$\mathbf{F}_{\text{rest}} = -\frac{e^2\,\mathbf{d}}{a_0^3} \equiv -m\omega_0^2\mathbf{d} \tag{5.1.7}$$

where $m$ is the mass of the electron, and this defines $\omega_0 = \sqrt{e^2/ma_0^3}$, which has units of frequency.

Also, in general, we can imagine that there is a damping force on the electron when it is in motion,

$$\mathbf{F}_{\text{damp}} = -m\gamma\,\dot{\mathbf{d}}, \qquad \text{where } \dot{\mathbf{d}} = \frac{d\mathbf{d}}{dt} \text{ is the velocity of the electron and } \gamma \text{ is the damping coefficient} \tag{5.1.8}$$

Physically, this damping is due to the coupling of the electron to other degrees of freedom in the system, often sound modes (phonons), to which it transfers energy.

In an external electric field $\mathbf{E}(t)$, the electron also experiences the force $\mathbf{F}_\mathbf{E} = -e\mathbf{E}(t)$.

The equation of motion for the electron displacement $\mathbf{d}$ is then,

$$m\ddot{\mathbf{d}} = \mathbf{F}_{\text{tot}} = \mathbf{F}_\mathbf{E} + \mathbf{F}_{\text{rest}} + \mathbf{F}_{\text{damp}} \qquad \text{where } \ddot{\mathbf{d}} = \frac{d^2\mathbf{d}}{dt^2} \text{ is the electron acceleration} \tag{5.1.9}$$

$$\Rightarrow \qquad \ddot{\mathbf{d}} + \gamma\dot{\mathbf{d}} + \omega_0^2\mathbf{d} = -\frac{e\mathbf{E}(t)}{m} \tag{5.1.10}$$

In the above, we assumed that $\mathbf{E}(t)$ was spatially constant over the atomic length scales on which the electron will move. This is usually a good assumption if $\mathbf{E}(t)$ is the field from an electromagnetic wave and $\lambda \gg a_0$, as is true for visible light.

If the electric field is constant in time, then after the electron responds it will settle into a fixed displacement with $\ddot{\mathbf{d}} = \dot{\mathbf{d}} = 0$, and so $\mathbf{d} = -e\mathbf{E}/m\omega_0^2 = -\mathbf{E}a_0^3/e$ and the dipole moment is $\mathbf{p} = -e\mathbf{d} = \mathbf{E}a_0^3$, as we found when we discussed the Clausius-Mossotti equation.

But now assume that the electric field is oscillating in time with frequency $\omega$,

$$\mathbf{E}(t) = \mathbf{E}_0 \mathrm{e}^{-i\omega t} \tag{5.1.11}$$

We leave it as understood that the physical field is given by the real part of this complex valued expression.

Now we want to solve the equation of motion (5.1.10) for the electron's displacement in this oscillating field. We will assume that the displacement also oscillates with the same frequency,

$$\mathbf{d}(t) = \mathbf{d}_0 \mathrm{e}^{-i\omega t} \tag{5.1.12}$$

and then substitute this form into (5.1.10) to get,

$$\left(-\omega^2 - i\omega\gamma + \omega_0^2\right) \mathbf{d}_0 = -\frac{e\mathbf{E}_0}{m} \tag{5.1.13}$$

so that the amplitude $\mathbf{d}_0$ of the displacement oscillations is

$$\mathbf{d}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \mathbf{E}_0 \tag{5.1.14}$$

The dipole moment on the atom is then,

$$\mathbf{p}(t) = -e\mathbf{d}(t) = \mathbf{p}_0 \mathrm{e}^{-i\omega t} \tag{5.1.15}$$

with

$$\mathbf{p}_0 = -e\mathbf{d}_0 = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \mathbf{E}_0 \equiv \alpha(\omega)\, \mathbf{E}_0 \tag{5.1.16}$$

and

$$\boxed{\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}} \qquad \text{is the } \textit{frequency dependent} \text{ atomic polarizability} \tag{5.1.17}$$

Of course the physical dipole moment must be real valued, so at the end we should remember to take the real part of the complex valued expression, to get $\mathbf{p}(t) = \mathrm{Re}[\alpha(\omega)\mathbf{E}_0 \mathrm{e}^{-i\omega t}]$.

Since $\alpha(\omega)$ is *complex* valued, the polarization does not in general oscillate in phase with $\mathbf{E}$. If we write $\alpha = \alpha_1 + i\alpha_2$, with $\alpha_1$ and $\alpha_2$ the real and imaginary parts of $\alpha$, then we can write,

$$\alpha(\omega) = |\alpha| \mathrm{e}^{i\delta}, \qquad \text{where } |\alpha| = \sqrt{\alpha_1^2 + \alpha_2^2} \text{ is the magnitude and } \delta = \arctan(\alpha_2/\alpha_1) \text{ is the phase of } \alpha \tag{5.1.18}$$

then we have,

$$\mathbf{p}(t) = \mathbf{p}_0 \mathrm{e}^{-i\omega t} = \alpha(\omega)\mathbf{E}_0 \mathrm{e}^{-i\omega t} = |\alpha| \mathrm{e}^{i\delta} \mathbf{E}_0 \mathrm{e}^{-i\omega t} = |\alpha| \mathbf{E}_0 \mathrm{e}^{-i(\omega t - \delta)} \tag{5.1.19}$$

so the oscillations of $\mathbf{p}(t)$ are shifted in phase by $\delta$ compared to the oscillations of $\mathbf{E}$.

Note, the above calculation is exactly the same as that of the forced, damped, harmonic oscillator, that you hopefully have seen in mechanics.

General time dependent response

The above was for an $\mathbf{E}$ field oscillating at a single frequency $\omega$. For a general time dependent field $\mathbf{E}(t)$, we can write $\mathbf{E}(t)$ in terms of its Fourier transform,

$$\mathbf{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{E}_\omega \, \mathrm{e}^{-i\omega t} \qquad \text{where } \mathbf{E}_\omega^* = \mathbf{E}_{-\omega} \text{ so that } \mathbf{E} \text{ is real valued.} \tag{5.1.20}$$

By superposition, the dipole response to the component $\mathbf{E}_\omega$ is just $\mathbf{p}_\omega = \alpha(\omega)\mathbf{E}_\omega$, and so

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{p}_\omega \, \mathrm{e}^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega)\mathbf{E}_\omega \, \mathrm{e}^{-i\omega t} \tag{5.1.21}$$

Substitute in the inverse Fourier transform for $\mathbf{E}_\omega$,

$$\mathbf{E}_\omega = \int_{-\infty}^{\infty} dt'\, \mathbf{E}(t')\, \mathrm{e}^{i\omega t} \tag{5.1.22}$$

and switch the order of integration to get,

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} dt'\, \mathbf{E}(t') \int_{-\omega}^{\omega} \frac{d\omega}{2\pi}\, \alpha(\omega)\, \mathrm{e}^{-i\omega(t-t')} = \int_{-\infty}^{\infty} dt'\, \mathbf{E}(t')\, \tilde{\alpha}(t-t') \tag{5.1.23}$$

where $\tilde{\alpha}(t) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \alpha(\omega)\, \mathrm{e}^{-i\omega t}$ is the Fourier transform of $\alpha(\omega)$.

So the dipole moment $\mathbf{p}(t)$ at time $t$ is due to the electric field $\mathbf{E}(t')$ at all times $t'$. The relation between $\mathbf{p}$ and $\mathbf{E}$ is *non-local* in time.

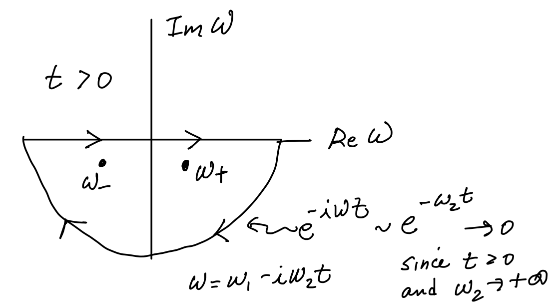
From a physical point of view, $\tilde{\alpha}(t)$ is the dipole response to an electric field $\mathbf{E}(t) = \mathbf{E}_0\delta(t)$. For such an *impulse* electric field, substituting into Eq. (5.1.23) gives $\mathbf{p}(t) = \tilde{\alpha}(t)\mathbf{E}_0$. From the *causal* point of view, we would expect that $\tilde{\alpha}(t) = 0$ for $t < 0$, i.e., there is no response in $\mathbf{p}$ *until* the electric field is turned on at $t = 0$. We will now see that our simple model for $\alpha(\omega)$ in Eq. (5.1.17) does indeed satisfy this expected behavior.

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\alpha(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{(-1)}{(\omega-\omega_+)(\omega-\omega_-)} \tag{5.1.24}$$

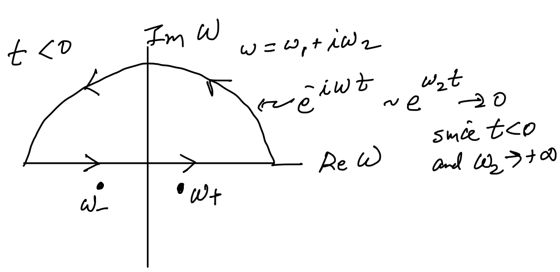
where

$$\omega_\pm = \frac{-i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} \quad = \quad \frac{-i\gamma}{2} \pm \bar{\omega}, \qquad \text{where } \bar{\omega} \equiv \sqrt{\omega_0^2 - \gamma^2/4} \tag{5.1.25}$$

The poles of the integrand of Eq. (5.1.24) are at $\omega = \omega_\pm = \dfrac{-i\gamma}{2} \pm \bar{\omega}$ which lie in the lower half of the complex plane. We can therefore do the integral using complex contour integration.




For $t > 0$, we continue the integral along the real axis to include the semicircular part in the lower half of the complex plane, so as to close the contour integration path, as shown on the left. On the closing semicircular part of the integration contour, the complex exponential piece in the integrand goes as $\mathrm{e}^{-i\omega t}$, where the complex $\omega = \omega_1 - i\omega_2$, and $\omega_2 \to +\infty$. Substituting in, we get $\mathrm{e}^{-i\omega t} = \mathrm{e}^{-i\omega_1 t}\mathrm{e}^{-\omega_2 t} \to 0$, since $\omega_2 \to +\infty$ and $t > 0$. So this semicircular piece gives zero, and the integral along the real axis is the same as the integral over the contour. We can therefore evaluate the contour integral using the Cauchy residue theorem. We will get a finite result for $\tilde{\alpha}(t)$ because the contour encloses the two poles at $\omega = \omega_\pm$.




But for $t < 0$, we continue the integral along the real axis to include the semicircular part in the upper half of the complex plane, so as to close the contour integration path as shown on the left. On the closing semicircular part of the integration contour, the complex exponential piece in the integrand goes as $\mathrm{e}^{-i\omega t}$, where the complex $\omega = \omega_1 + i\omega_2$, and $\omega_2 \to +\infty$. Substituting in, we get $\mathrm{e}^{-i\omega t} = \mathrm{e}^{-i\omega_1 t}\mathrm{e}^{\omega_2 t} \to 0$, since $\omega_2 \to +\infty$ and $t < 0$. So this semicircular piece gives zero, and the integral along the real axis is the same as the integral over the contour. We can therefore evaluate the contour integral using the Cauchy residue theorem. But since this contour encloses no poles of the integrand, the Cauchy theorem gives that this integral is zero. Hence $\tilde{\alpha}(t) = 0$ for $t < 0$, and the response function $\tilde{\alpha}(t)$ is indeed causal.

We can now evaluate $\tilde{\alpha}(t)$ for $t > 0$. The Cauchy theorem says that the contour integral is $2\pi i$ times the sum of the residues enclosed by the contour, if one goes around the contour counterclockwise, and $-2\pi i$ times the sum of the residues, if one goes around the contour clockwise. If $f(z)$ is the integrand with a simple pole at $z_0$, then the residue at $z_0$ is $\lim_{z\to z_0}(z - z_0)f(x)$.

For $t > 0$ we close the contour in the lower half of the complex plane and therefore we go around clockwise. The residues are at $\omega = \omega_\pm = -i\gamma/2 \pm \bar{\omega}$. We therefore have,

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{(-1)}{(\omega-\omega_+)(\omega-\omega_-)} = (-2\pi i)\, \frac{e^2}{m}\, \frac{(-1)}{2\pi} \left[\frac{\mathrm{e}^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{\mathrm{e}^{-i\omega_- t}}{\omega_- \omega_+}\right] \tag{5.1.26}$$

where the first term in the brackets is from the pole at $\omega_+$, and the second term is from the pole at $\omega_-$.

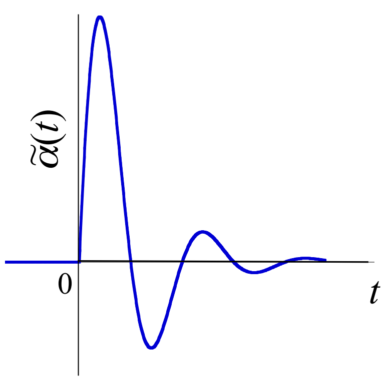
Substituting in for $\omega_\pm$ we then get

$$\tilde{\alpha}(t) = \frac{ie^2}{m}\left[\frac{\mathrm{e}^{-\gamma t/2}\,\mathrm{e}^{-i\bar{\omega}t}}{2\bar{\omega}} - \frac{\mathrm{e}^{-\gamma t/2}\,\mathrm{e}^{i\bar{\omega}t}}{2\bar{\omega}}\right] \tag{5.1.27}$$

Using the complex exponential form, $\sin x = \dfrac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{2i}$, we then have,

$$\tilde{\alpha}(t) = \begin{cases} \dfrac{e^2}{m\bar{\omega}}\, \mathrm{e}^{-\gamma t/2}\, \sin(\bar{\omega}t) & t > 0 \\ 0 & t < 0 \end{cases} \tag{5.1.28}$$

We thus see that the dipole response to an impulse electric field $\mathbf{E}(t) = \delta(t)$ is a damped oscillation. This is the same response as for a damped harmonic oscillator that is given a sudden kick.




**Dielectric Materials**

The above discussion was for an individual atom. For the macroscopic response of a material composed of such atoms we expect the polarization density is

$$\mathbf{P}_\omega = 4\pi\chi_e(\omega)\mathbf{E}_\omega \tag{5.1.29}$$

for a harmonically oscillating field. For a dilute system we can take as a simple approximation, $\chi_e(\omega) \approx n\alpha(\omega)$, with $n$ the density of polarizable atoms (we can use the Clausius-Mossoti correction for denser materials).

Therefore we can write for the electric displacement field,

$$\mathbf{D}_\omega = \epsilon(\omega)\mathbf{E}_\omega \qquad \text{where} \qquad \epsilon(\omega) = 1 + 4\pi\chi_e(\omega) \tag{5.1.30}$$

The consequence of this is that, just as with the relation between $\mathbf{p}$ and $\mathbf{E}$, the relation between $\mathbf{D}$ and $\mathbf{E}$ is *non-local* in time,

$$\mathbf{D}(t) \neq \epsilon\mathbf{E}(t) \quad \text{but rather} \quad \mathbf{D}(t) = \int_{-\infty}^{\infty} dt'\, \mathbf{E}(t')\tilde{\epsilon}(t-t') \tag{5.1.31}$$

where $\tilde{\epsilon}(t)$ is the Fourier transform of $\epsilon(\omega)$.

Ampere's Law, $\boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$, if we write it in terms of $\mathbf{E}$ and $\mathbf{B}$, becomes,

$$\frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\int_{-\infty}^{\infty} dt'\,\mathbf{E}(t')\,\frac{d\tilde{\epsilon}(t-t')}{dt} \tag{5.1.32}$$

an *integro-differential* equation! Here we took $\mu$ as a constant. In principle $\mu$ can also vary with frequency, but for simplicity we will take it as a constant in these notes.

**Harmonic Plane Waves in a Dielectric**

The moral of the story is: the macroscopic Maxwell's equations in linear materials only look simple when expressed in terms of the Fourier transforms of the fields,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathbf{E}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad & \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ \mathbf{D}(\mathbf{r},t) &= \mathbf{D}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad & \mathbf{H}(\mathbf{r},t) &= \mathbf{H}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \end{aligned} \tag{5.1.33}$$

Then, $\mathbf{D}_\omega = \epsilon(\omega)\mathbf{E}_\omega$ and $\mathbf{B}_\omega = \mu\mathbf{H}$, and the macroscopic Maxwell's equations for a source free system, $\rho = \mathbf{j} = 0$,

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \tag{5.1.34}$$

become

$$\begin{aligned} &1) \quad i\mathbf{k}\cdot\mathbf{D}_\omega = i\epsilon(\omega)\mathbf{k}\cdot\mathbf{E}_\omega = 0 \qquad &&\Rightarrow \quad \mathbf{k}\perp\mathbf{E}_\omega \quad \text{unless } \epsilon(\omega)=0 \\ &2) \quad i\mathbf{k}\cdot\mathbf{B} = 0 &&\Rightarrow \quad \mathbf{k}\perp\mathbf{B}_\omega \\ &3) \quad i\mathbf{k}\times\mathbf{E}_\omega = \frac{i\omega}{c}\mathbf{B}_\omega && \\ &4) \quad i\mathbf{k}\times\mathbf{H}_\omega = \frac{-i\omega}{c}\mathbf{D}_\omega &&\Rightarrow \quad \frac{i\mathbf{k}}{\mu}\times\mathbf{B}_\omega = \frac{-i\omega}{c}\epsilon(\omega)\mathbf{E}_\omega \end{aligned} \tag{5.1.35}$$

$$\mathbf{k}\times(3) \quad\Rightarrow\quad i\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_\omega) = \frac{i\omega}{c}\mathbf{k}\times\mathbf{B}_\omega \quad\Rightarrow\quad i\mathbf{k}(\mathbf{k}\cdot\mathbf{E}_\omega) - i\mathbf{E}_\omega(\mathbf{k}\cdot\mathbf{k}) = \frac{i\omega}{c}\left(\frac{-\omega\mu}{c}\epsilon(\omega)\mathbf{E}_\omega\right) \tag{5.1.36}$$

where we used Ampere's law (4) in the last step. Then using $\mathbf{k}\cdot\mathbf{E}_\omega = 0$ from (1), we get,

$$-ik^2\mathbf{E}_\omega = \frac{-i\omega^2}{c^2}\epsilon(\omega)\mu\mathbf{E}_\omega \quad\Rightarrow\quad \boxed{k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)\mu} \quad \text{dispersion relation for waves in a dielectric} \tag{5.1.37}$$

Note, $\dfrac{\omega}{|\mathbf{k}|} = \dfrac{c}{\sqrt{\epsilon(\omega)\mu}}$ varies with $\omega$, so there is not a single phase velocity.

$\Rightarrow$ $\mathbf{E}$ is *not* in general a solution of a wave equation – different frequencies travel with different speeds!

**Complex, Frequency Dependent, Dielectric Function**

From our result, $\epsilon(\omega) = 1 + 4\pi\chi_e(\omega) \approx 1 + 4\pi n\alpha(\omega)$, we see that $\epsilon(\omega)$ is a frequency dependent, complex valued, function. We can write it in terms of its real and imaginary parts, $\epsilon_1$ and $\epsilon_2$,

$$\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega) \qquad \text{where } \epsilon_1 \text{ and } \epsilon_2 \text{ are both real valued.} \tag{5.1.38}$$

Let us take a wave traveling in the $\hat{\mathbf{z}}$ direction, $\mathbf{k} = k\hat{\mathbf{z}}$. Because $\epsilon$ is complex valued, then by the dispersion relation $k$ will also be complex valued, $k = k_1 + ik_2$, where $k_1$ and $k_2$ are real valued. From Eq. (5.1.37) we get,

$$k = k_1 + ik_2 = \pm\frac{\omega}{c}\sqrt{\mu}\sqrt{\epsilon_1 + i\epsilon_2} \tag{5.1.39}$$

The resulting wave is,

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=\mathbf{E}_\omega\,\mathrm{e}^{i[(k_1+ik_2)z-\omega t]}=\mathbf{E}_\omega\,\mathrm{e}^{-k_2 z}\;\mathrm{e}^{i(k_1 z-\omega t)} \tag{5.1.40}$$

We see that $k_1$ determines the oscillation of the wave, while $k_2$ determines the decay or *attenuation* of the wave as it propagates into the material. What $k_2$ represents is the transfer of energy from the wave into the material; in our simple model this arises because of the damping term $\mathbf{F}_{\rm damp}$ that is in the equation of motion for the polarizing atom.

We have the following definitions.

phase velocity: $v_p\equiv\dfrac{\omega}{k_1}$ speed with which the peaks of oscillation move

index of refraction: $n\equiv\dfrac{c}{v_p}=\dfrac{ck_1}{\omega}$

group velocity: $v_g$ defined by $\dfrac{1}{v_g}=\dfrac{dk_1}{d\omega}$ speed with which the envelope of a wave pulse moves

You have presumably already seen the group velocity in quantum mechanics, as the speed with which a particle represented by a wave packet moves.

For the wave in the dielectric with $\mathbf{k}=k\hat{\mathbf{z}}$, the magnetic field amplitude is given by Faraday's law (3) as,

$$\mathbf{B}_\omega=\frac{c\mathbf{k}}{\omega}\times\mathbf{E}_\omega=\frac{c(k_1+ik_2)}{\omega}\,\hat{\mathbf{z}}\times\mathbf{E}_\omega \tag{5.1.41}$$

If we write the complex $k$ in terms of its amplitude and phase, $k=|k|\mathrm{e}^{i\delta}$, with $|k|=\sqrt{k_1^2+k_2^2}$ and $\delta=\arctan(k_2/k_1)$, then we have,

$$\mathbf{B}_\omega=\frac{c|k|}{\omega}\,\hat{\mathbf{z}}\times\mathbf{E}_\omega\,\mathrm{e}^{i\delta} \tag{5.1.42}$$

so

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_\omega\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=\frac{c|k|}{\omega}\,(\hat{\mathbf{z}}\times\mathbf{E}_\omega)\,\mathrm{e}^{-k_2 z}\;\mathrm{e}^{i(k_1 z-\omega t+\delta)} \tag{5.1.43}$$

Taking the real part of the complex exponential representation, we get for the physical fields,

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_\omega\,\mathrm{e}^{-k_2 z}\,\cos(k_1 z-\omega t) \tag{5.1.44}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{c|k|}{\omega}\,(\hat{\mathbf{z}}\times\mathbf{E}_\omega)\,\mathrm{e}^{-k_2 z}\,\cos(k_1 z-\omega t+\delta) \tag{5.1.45}$$

We see that the oscillations in $\mathbf{B}$ are shifted in phase by an amount $\delta$ from the oscillations in $\mathbf{E}$.

Summary:

1) $\mathbf{E}$ and $\mathbf{B}\perp\mathbf{k}$ *transverse polarized wave*: fields are in directions perpendicular to $\mathbf{k}$

2) $\mathbf{E}\perp\mathbf{B}$

3) $\dfrac{|\mathbf{B}|}{|\mathbf{E}|}=\dfrac{c|k|}{\omega}=\sqrt{|\epsilon(\omega)|\mu}$ *amplitude ratio* (5.1.46)

4) $\mathbf{B}$ is shifted in phase with respect to $\mathbf{E}$ by $\delta=\arctan(k_2/k_1)$

5) waves decay as they propagate, $\sim\mathrm{e}^{-k_2 z}$

Items (4) and (5) are consequences of the fact that $\epsilon(\omega)$ is a *complex valued* quantity. If $\epsilon_2 = 0$, i.e., $\epsilon(\omega)$ is real valued, then $k_2 = 0$ and there is no decay and no phase shift.

6) $\mathbf{E}(t)$ and $\mathbf{D}(t)$ are non-locally related in time

7) waves of different $\omega$ travel with different $v_p = \omega/k_1$ (5.1.47)

8) <u>dispersion</u>: wave *pulses* do *not* travel with $v_p$, but rather travel with the speed $v_g = d\omega/dk$

Items (6) – (8) are consequences of the fact that $\epsilon(\omega)$ varies with $\omega$.

The term *dispersion* refers to the fact that waves of different $\omega$ travel with different phase velocities, and so the phase velocity $v_p$ and the group velocity $v_g$ are not the same. It happens whenever the *dispersion relation* between $\omega$ and $k$ is *non-linear.* As you know from quantum mechanics, the speed of a finite time wave pulse (as opposed to an infinite harmonic plane wave) travels with $v_g$. Also, the width of that wave pulse spreads as the wave pulse propagates.

When we have $v_g < v_p$, this is referred to as *normal dispersion.*

When we have $v_g > v_p$, this is referred to as *anomalous dispersion.*

This is often expressed in terms of the behavior of the index of refraction $n$. To see this,

$$\frac{1}{v_g} \equiv \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c}n\right] \qquad \text{since } k_1 = \omega/v_p \text{ and } v_p = c/n \tag{5.1.48}$$

$$= \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega} \tag{5.1.49}$$

$$\Rightarrow \qquad v_g = \frac{v_p}{1 + \dfrac{v_p\,\omega}{c}\dfrac{dn}{d\omega}} \tag{5.1.50}$$

So when

$$\frac{dn}{d\omega} > 0, \qquad v_g < v_p \qquad \text{normal dispersion} \tag{5.1.51}$$

$$\frac{dn}{d\omega} < 0, \qquad v_g > v_p \qquad \text{anomalous dispersion} \tag{5.1.52}$$

Normal dispersion is when the index of refraction is an increasing function of frequency, anomalous dispersion is when the index of refraction is a decreasing function of frequency.