1) **Discussion Question 1.4.2**

In the Coulomb gauge, we had for the scalar potential,

$$\phi(\mathbf{r},t) = \int d^3r' \, \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

so that the potential at point $\mathbf{r}$ at time $t$ is effected by the charge density some distance away at $\mathbf{r}'$ at the exact same time $t$. It would seem that information about the charge is transmitted across the distance $|\mathbf{r}-\mathbf{r}'|$ *instantaneously*, which would violate our notion of physical causality: since no information can travel faster than the speed of light $c$, it should take some finite amount of time $\Delta t \geq |\mathbf{r}-\mathbf{r}'|/c$ for a charge at $\mathbf{r}'$ to effect the potential some distance away at $\mathbf{r}$. So how can the Eq. (1) above be correct?

The answer is given by noting that $\phi(\mathbf{r},t)$ is not itself directly measurable, in a non-static situation. The only things measurable are the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. The electric field, in a non-static situation, is given by,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{2}$$

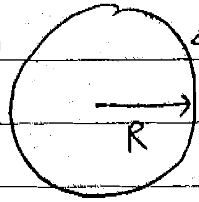
The term $-\boldsymbol{\nabla}\phi$ can be non-causal *if* causality in $\mathbf{E}$ is restored by the $\partial\mathbf{A}/\partial t$ term.

In the Coulomb gauge we saw that the equation for $\mathbf{A}$ is given by, $\Box^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}_\perp$, where $\mathbf{j}_\perp$ is the transverse part of the current density $\mathbf{j}$, and is given by,

$$\mathbf{j}_\perp(\mathbf{r},t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\left[\int d^3r' \, \frac{\boldsymbol{\nabla}'\times\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{3}$$

Since the definition of $\mathbf{j}_\perp$ involves an integral over all space of $\mathbf{j}$, then $\mathbf{j}_\perp$ is not *locally* related to $\mathbf{j}$, and so there is no causal relation between the two, i.e., $\mathbf{j}_\perp(\mathbf{r},t)$ is instantaneously determined by $\mathbf{j}(\mathbf{r}',t)$ a finite distance $|\mathbf{r}-\mathbf{r}'|$ away. Hence $\mathbf{j}_\perp$, and so also $\mathbf{A}$ which is determined by $\mathbf{j}_\perp$, are *not* causally related to the sources $\rho$ and $\mathbf{j}$.

Thus, not only is the $-\boldsymbol{\nabla}\phi$ term in the expression for $\mathbf{E}$ non-causal, but so also is the $\partial\mathbf{A}/\partial t$ term. It can be shown (I won't do it) that the sum of the two terms gives rise to a perfectly causal relation between $\mathbf{E}$ and the sources $\rho$ and $\mathbf{j}$. One way to know that this must be so is to note that in the Lorenz gauge the potentials $\phi$ and $\mathbf{A}$ are causally related to the sources $\rho$ and $\mathbf{j}$, since they solve the inhomogeneous wave equation. And since $\phi$ and $\mathbf{A}$ are causal, then so are $\mathbf{E}$ and $\mathbf{B}$ since they are locally determined from $\phi$ and $\mathbf{A}$. And since $\mathbf{E}$ is clearly causal in the Lorenz gauge, it must also be so in the Coulomb gauge, since the fields $\mathbf{E}$ and $\mathbf{B}$ are independent of whatever gauge one chooses to work in.

2) a) 


← surface charge uniform $\sigma = \frac{Q}{4\pi R^2}$

in spherical coordinates

$$\rho(r,\theta,\varphi) = C\,\delta(r-R)$$

restricts $\rho$ to shell at radius $r=R$

independent of $\theta$ and $\varphi$ by symmetry

Now the total charge $Q$ is the integral of $\rho$, hence

$$Q = \int_0^\infty dr\, r^2 \sin\theta \int_0^\pi d\theta \int_0^{2\pi} d\varphi\; \rho(r,\theta,\varphi)$$

$$= 4\pi \int_0^\infty dr\, r^2\, C\,\delta(r-R) = 4\pi R^2 C$$

allows us to determine $C = \frac{Q}{4\pi R^2} = \sigma$

$$\boxed{\rho(r,\theta,\varphi) = \frac{Q}{4\pi R^2}\,\delta(r-R)}$$

b) 


← uniform surface charge $\sigma = \frac{\lambda}{2\pi R}$

in cylindrical coords

$$\rho(r,\varphi,z) = C\;\delta(r-R)$$

cylindrical radial coord

restricts $\rho$ to surface of cylinder at $r=R$

independent of $\varphi$ and $z$ by symmetry

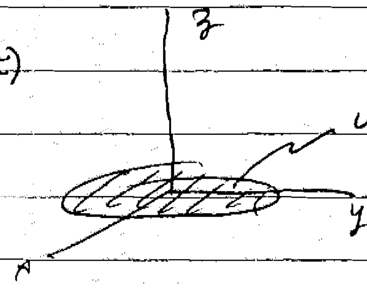
$$Q = \int_0^\infty dr\, r \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz \;\; \rho(r, \varphi, z)$$

$$= 2\pi \int_{-\infty}^{\infty} dz \int_0^\infty dr\, r \;\; c\, \delta(r-R)$$

$$= 2\pi \int_{-\infty}^{\infty} dz \;\; cR \;\; = \;\; \lambda \int_{-\infty}^{\infty} dz$$

$$\Rightarrow \quad C = \frac{\lambda}{2\pi R} \qquad \boxed{\rho(r, \varphi, z) = \frac{\lambda}{2\pi R}\, \delta(r-R)}$$

c)




uniform surface charge $\sigma = \dfrac{Q}{\pi R^2}$

in cylindrical coords

$$\rho(r, \varphi, z) = C(r)\, \delta(z)$$

$\delta(z)$ restricts $\rho$ to $z=0$ plane

$C(r)$: indep of $\varphi$ by symmetry; may depend on $r$

Total charge within radii $r$ to $r+dr$ $(r<R)$ is

$$dq = dr\, r \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi \;\; \rho\,(r, \varphi, z)$$

$$= 2\pi\, C(r)\, r\, dr \qquad \text{(do integral)}$$

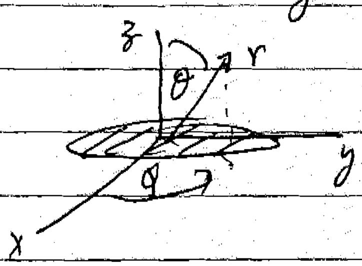
$$= \sigma\, 2\pi r\, dr \qquad \text{(since charge is uniform } \sigma \text{ on di}$$

$$\Rightarrow \quad C(r) = \sigma = Q/\pi R^2 \quad \text{for } r<R$$

and clearly $C(r) = 0$ for $r > R$

$$\boxed{\rho(r,\varphi,z) = \begin{cases} \dfrac{Q}{\pi R^2}\,\delta(z) & r<R \\ 0 & r>R \end{cases}}$$

d) same as (c) but in ~~cylindrical~~ spherical coords




<u>method I</u>:

$$\rho(r,\theta,\varphi) = C(r)\,\delta(\theta - \pi/2)$$

restricts $\rho$ to the $z=0$ plane

indep of $\varphi$ by symmetry
can depend on $r$

as in part (c) the charge $dq$ between radii $r$ to $r+dr$ is $(r<R)$

$$dq = r^2 dr \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi\; \rho(r,\theta,\varphi)$$

$$= 2\pi r^2 dr\, C(r) \int_0^{\pi} d\theta \sin\theta\; \delta(\theta - \pi/2)$$

$$= 2\pi r^2 C(r)\, dr$$

$$= \sigma\, 2\pi r\, dr$$ (same as in part (c) since charge is uniform on disk

$$\Rightarrow C(r) = \frac{\sigma}{r} = \frac{Q}{\pi R^2 r}$$

$$\boxed{\rho(r,\theta,\varphi) = \begin{cases} \dfrac{Q}{\pi R^2 r}\,\delta(\theta - \pi/2) & r < R \\ 0 & r > R \end{cases}}$$

Note: dimensions of $\rho$ must be charge/vol
Since $\delta(\theta-\pi/2)$ is dimensionless (since $\theta$ is)
the prefactor must have dimensions 1/vol, which it does.

In cylindrical coords however, $\delta(z)$ has units 1/length, so the prefactor must have dimensions 1/area, which it does.

<u>Method II</u>: Use result in (c), $\rho = \frac{Q}{\pi R^2}\delta(z)$
and convert to spherical coords

in spherical coords, $z = r\cos\theta$
For fixed $r \neq 0$
$$\delta(z) = \delta(r\cos\theta) = \frac{1}{r}\,\delta(\cos\theta)$$

which follows from general rule:
$$\delta(ax) = \frac{1}{a}\,\delta(x)$$

change variables $y = ax$

proof:
$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(ax) = \int_{-\infty}^{\infty} \frac{dy}{a}\, f(y/a)\,\delta(y)$$
$$= \frac{1}{a} f(0) = \frac{1}{a}\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x)$$

Next use general result

$$\delta(g(x)) = \frac{1}{\left|\frac{dg}{dx}\right|_{x=x_0}} \delta(x - x_0) \quad \text{where } g(x_0) = 0$$

$x_0$ is zero of $g$

proof: similar to last proof

change variables $y = g(x)$

$$\int_{x_1}^{x_2} dx\, f(x)\, \delta(g(x)) = \int_{g(x_1)}^{g(x_2)} \frac{dy}{\left(\frac{dg}{dx}\right)} f(g^{-1}(y))\, \delta(y)$$

assume $x_1 < x_0 < x_2$

$$= \frac{f(x_0)}{\left|\frac{dg}{dx}\right|_{x=x_0}}$$

absolute value signs since if $\left.\frac{dg}{dx}\right|_{x=x_0} < 0$, we have to reverse the limits of integration, which adds a (−) sign.

So $$\delta(\cos\theta) = \frac{\delta(\theta - \pi/2)}{\sin(\pi/2)} = \delta(\theta - \pi/2)$$

So finally $$\delta(z) = \frac{\delta(\theta - \pi/2)}{r}$$ and so

$$\rho = \begin{cases} \dfrac{Q}{\pi R^2} \dfrac{\delta(\theta - \pi/2)}{r} & r < R \\ 0 & r > R \end{cases}$$

as found by method I

3) $\phi = q\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$ $\qquad \frac{1}{\alpha} = \frac{a_0}{2}$

to get the charge density $\rho$ use

$$-\nabla^2\phi = 4\pi\rho$$

In doing this calculation we have to watch out for terms like $\nabla^2\left(\frac{1}{r}\right)$ since we know this will give a delta function. We know this must somehow be there since for small r,

$$\phi(r) \sim \frac{q}{r}\left[1 - \frac{\alpha r}{2} + \cdots\right] \quad \text{(Taylor series)}$$

will give $\delta(r)$

Now:

$$\phi = q\frac{e^{-\alpha r}}{r} + \frac{\alpha}{2}q\,e^{-\alpha r}$$

no $\frac{1}{r}$ pieces so safe to take $\nabla^2$

2nd piece:

$$\nabla^2\left(\frac{\alpha}{2}q\,e^{-\alpha r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{\alpha}{2}q\,e^{-\alpha r}\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{\alpha}{2}q\,(-\alpha)e^{-\alpha r}\right)$$

$$= -\frac{\alpha^2 q}{2}\frac{1}{r^2}\left[2re^{-\alpha r} - \alpha r^2 e^{-\alpha r}\right]$$

$$= \frac{\alpha^2 q}{2}\left[\alpha - \frac{2}{r}\right]e^{-\alpha r}$$ contributes to $4\pi\rho$

for the 1st piece we use, for any scalar $\varphi$ and $\psi$

$$\nabla^2(\varphi\psi) = \vec{\nabla}\cdot\vec{\nabla}(\varphi\psi) = \vec{\nabla}\cdot\left(\psi\vec{\nabla}\varphi + \varphi\vec{\nabla}\psi\right)$$
$$= (\vec{\nabla}\psi)\cdot(\vec{\nabla}\varphi) + \psi\nabla^2\varphi + (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\psi) + \varphi\nabla^2\psi$$
$$= \psi\nabla^2\varphi + 2(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\psi) + \varphi\nabla^2\psi$$

apply to 1st piece with $\varphi = \frac{1}{r}$, $\psi = q e^{-\alpha r}$

$$\nabla^2\varphi = -4\pi\delta(\vec{r}) \qquad \vec{\nabla}\varphi = \hat{r}\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

$$\vec{\nabla}\psi = \hat{r}\frac{d}{dr}\left(q e^{-\alpha r}\right) = -\alpha q e^{-\alpha r}\,\hat{r}$$

$$\nabla^2\psi = \frac{-1}{r^2}\frac{d}{dr}\left(r^2\alpha q e^{-\alpha r}\right) = \frac{-\alpha q}{r^2}\left[2r - \alpha r^2\right]e^{-\alpha r}$$
$$= \alpha q\left[\alpha - \frac{2}{r}\right]e^{-\alpha r}$$

So

$$\nabla^2\left(\frac{q e^{-\alpha r}}{r}\right) = -4\pi\delta(\vec{r})\, q e^{-\alpha r} + 2\left(\frac{-\hat{r}}{r^2}\right)\left(-\alpha q e^{-\alpha r}\hat{r}\right)$$

can set $e^{-\alpha r}=1$ since only not zero at $r=0$

$$+ \alpha q\left[\frac{\alpha}{r} - \frac{2}{r^2}\right]e^{-\alpha r}$$

Adding both pieces together gives

$$\nabla^2\phi = -4\pi q\,\delta(\vec{r}) + q\left[\frac{2\alpha}{r^2} + \frac{\alpha^2}{r} - \frac{2\alpha}{r^2} + \frac{\alpha^3}{2} - \frac{\alpha^2}{r}\right]e^{-\alpha r}$$
$$= -4\pi q\,\delta(\vec{r}) + q\frac{\alpha^3}{2}e^{-\alpha r}$$
$$= 4\pi\rho$$

So

$$\boxed{\rho = q\,\delta(\vec{r}) - \frac{q\,\alpha^3}{8\pi}\, e^{-\alpha r}}$$

to see if this makes sence recall that electron wave function in ground state is

$$\psi(r) = \frac{1}{\sqrt{\pi}\, a_0^{3/2}}\, e^{-r/a_0}$$ where $a_0 = 2/\alpha$ is Bohr radius

So charge density from the electron of charge $-q$ is

$$q\,|\psi(r)|^2 = \frac{-q}{\pi a_0^3}\, e^{-r\,2/a_0} = \frac{-q\,\alpha^3}{8\pi}\, e^{-\alpha r}$$

So the $2^{nd}$ term in $\rho$ above is the charge density of the electron cloud, and the $1^{st}$ term in $\rho$ is the point charge $q$ of the proton at the origin.

4) a) $\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, $\phi = \phi_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

magnetic field is given by

$$\vec{B} = \vec{\nabla}\times\vec{A} = i\vec{k}\times\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \equiv \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\boxed{\vec{B}_0 = i\vec{k}\times\vec{A}_0}$$

electric field is given by

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = (-i\vec{k}\phi_0 + i\frac{\omega}{c}\vec{A}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\equiv \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\boxed{\vec{E}_0 = -i\vec{k}\phi_0 + i\frac{\omega}{c}\vec{A}_0 = -i\vec{k}\phi_0 + ik\vec{A}_0}$$

where for plane electromagnetic wave $\frac{\omega}{c} = k \equiv |\vec{k}|$

The requirement that the $\vec{E}$ and $\vec{B}$ above solve the source free ($\rho = 0$, $\vec{J} = 0$) maxwell equations will give the desired relation between $\phi_0$ and $\vec{A}_0$

1) $\vec{\nabla}\cdot\vec{B} = 0$ 2) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ Faraday

3) $\vec{\nabla}\cdot\vec{E} = 0$ Gauss 4) $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ Ampere

For $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

these become

1) $i\vec{k}\cdot\vec{B}_0 = 0$ 2) $i\vec{k}\times\vec{E}_0 = i\frac{\omega}{c}\vec{B}_0 = ik\vec{B}_0$

3) $i\vec{k}\cdot\vec{E}_0 = 0$ 4) $i\vec{k}\times\vec{B}_0 = -i\frac{\omega}{c}\vec{E}_0 = -ik\vec{E}_0$

Substitute in for $\vec{E}_0$ and $\vec{B}_0$ in terms of $\vec{A}_0$ and $\phi_0$

1) $\Rightarrow i\vec{k}\cdot(i\vec{k}\times\vec{A}_0) = 0$ automatically satisfied ✓

2) $i\vec{k}\times(-i\vec{k}\phi_0 + ik\vec{A}_0) = ik(i\vec{k}\times\vec{A}_0)$

$$\underbrace{i\vec{k}\times(-i\vec{k}\phi_0 + ik\vec{A}_0)}_{=\,-k\vec{k}\times\vec{A}_0} = -k\,\vec{k}\times\vec{A}_0$$

automatically satisfied ✓

3) $i\vec{k}\cdot(-i\vec{k}\phi_0 + ik\vec{A}_0) = 0$

$$\Rightarrow k^2\phi_0 = k\,\vec{k}\cdot\vec{A}_0 \Rightarrow \boxed{\phi_0 = \frac{\vec{k}\cdot\vec{A}_0}{k} = \hat{k}\cdot\vec{A}_0}$$

desired relation between $\phi_0$ and $\vec{A}_0$

4) $i\vec{k}\times(i\vec{k}\times\vec{A}_0) = -ik(-ik\phi_0 + ik\vec{A}_0)$

$$k^2\vec{A}_0 - \vec{k}\,\vec{k}\cdot\vec{A}_0 = -k^2\phi_0 + k^2\vec{A}_0$$

$$\Rightarrow \frac{\vec{k}\,\vec{k}\cdot\vec{A}_0}{k^2} = \phi_0 \quad \text{or} \quad \boxed{\phi_0 = \hat{k}\cdot\vec{A}_0}$$

same as for (3

b) Gauge transformation

$$\left.\begin{aligned}\vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}\end{aligned}\right\}$$ leaves $\vec{E}$ and $\vec{B}$ unchanged

$$\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We want $\vec{A}'$ transverse, ie $\vec{k}\cdot\vec{A}_0' = 0$

We can get this if we subtract off $\vec{A}$ its longitudinal part

$$\boxed{\vec{A}' = (\vec{A}_0 - \hat{k}\,\hat{k}\cdot\vec{A}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}} = \vec{A}_0' e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Then $\vec{k}\cdot\vec{A}_0' = 0$

$$\Rightarrow \vec{\nabla}\chi = \vec{A}' - \vec{A} = -\hat{k}\,\hat{k}\cdot\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= -\frac{\vec{k}\ \vec{k}\cdot\vec{A}_0}{k^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\Rightarrow \boxed{\chi = i\frac{\vec{k}\cdot\vec{A}_0}{k^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}}$$ take gradient of $\chi$ and check that get above

c) So $\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$

$$= \left(\phi_0 - \frac{\omega}{c}\frac{\vec{k}\cdot\vec{A}_0}{k^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= (\phi_0 - \hat{k}\cdot\vec{A}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$ as $\frac{\omega}{c} = k$, $\hat{k} = \vec{k}/k$

$$\phi' = (\phi_0 - \hat{k} \cdot \vec{A}_0) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

using result from (a) that $\phi_0 = \hat{k} \cdot \vec{A}_0$ we get

$$\boxed{\phi' = 0}$$

scalar potential vanishes in the gauge in which vector potential is transverse

5)

We have $\vec{f}(\vec{r}) = \vec{f}_{\parallel}(\vec{r}) + \vec{f}_{\perp}(\vec{r})$

where $\vec{f}_{\parallel}(\vec{r}) = \frac{-1}{4\pi} \vec{\nabla} \int d^3r' \frac{\vec{\nabla}' \cdot \vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

$\vec{f}_{\perp}(\vec{r}) = \frac{1}{4\pi} \vec{\nabla} \times \int d^3r' \frac{\vec{\nabla}' \times \vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

substitute into the above

$$\vec{f}(\vec{r}') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}'} \vec{f}(\vec{k})$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')} \frac{4\pi}{k'^2}$$

$$\vec{f}_{\parallel}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3r' \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')} \frac{4\pi}{k'^2} \vec{\nabla}' \cdot \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}'} \vec{f}(\vec{k})$$

Now use

$$\vec{\nabla}' \cdot \left(e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k})\right) = \left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}'}\right) \cdot \vec{f}(\vec{k}) = e^{i\vec{k}\cdot\vec{r}'} i\vec{k} \cdot \vec{f}(\vec{k})$$

and

$$\vec{\nabla} e^{i\vec{k}'\cdot\vec{r}} = i\vec{k}' e^{i\vec{k}'\cdot\vec{r}}$$

and rearrange order of integrations to get

$$\vec{f}_{\parallel}(\vec{r}) = -\frac{1}{4\pi} \int \frac{d^3k'}{(2\pi)^3} i\vec{k}' e^{i\vec{k}'\cdot\vec{r}} \frac{4\pi}{k'^2} \int \frac{d^3k}{(2\pi)^3} i\vec{k} \cdot \vec{f}(\vec{k})$$

$$\times \int d^3r' e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}$$

use Fourier transform of Dirac $\delta$-function

$$\int d^3r'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = (2\pi)^3\,\delta(\vec{k}-\vec{k}')$$

$$\vec{f}_{\parallel}(\vec{r}) = \int \frac{d^3k'}{(2\pi)^3}\, \frac{\vec{k}'}{k'^2}\, e^{i\vec{k}'\cdot\vec{r}} \int \frac{d^3k}{(2\pi)^3}\, \vec{k}\cdot\vec{f}(\vec{k})\, (2\pi)^3\,\delta(\vec{k}-\vec{k}')$$

do $\vec{k}'$ integration by using the $\delta$-function, sets $\vec{k}'=\vec{k}$

$$\vec{f}_{\parallel}(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, \frac{\vec{k}\ \vec{k}\cdot\vec{f}(\vec{k})}{k^2}\, e^{i\vec{k}\cdot\vec{r}}$$

From this we get at once that Fourier transf of $\vec{f}_{\parallel}(\vec{r})$ is

$$\vec{f}_{\parallel}(\vec{k}) = \frac{\vec{k}\ \vec{k}\cdot\vec{f}(\vec{k})}{k^2} = \hat{k}\ \hat{k}\cdot\vec{f}(\vec{k})$$

projection of $\vec{f}(\vec{k})$ onto direction $\hat{k}$

Similarly for $\vec{f}_{\perp}(\vec{r})$ except now use

$$\vec{\nabla}'\times\left(e^{i\vec{k}\cdot\vec{r}'}\,\vec{f}(\vec{k})\right) = \left(\vec{\nabla}' e^{i\vec{k}\cdot\vec{r}'}\right)\times\vec{f}(\vec{k})$$
$$e^{i\vec{k}\cdot\vec{r}'}\, i\vec{k}\times\vec{f}(\vec{k})$$

and

$$\vec{\nabla}\times\left(e^{i\vec{k}'\cdot\vec{r}}\,\vec{A}\right) = e^{i\vec{k}'\cdot\vec{r}}\, i\vec{k}'\times\vec{A}$$

any const vector $\vec{A}$

$$\vec{f}_{\perp}(\vec{r}) = \frac{1}{4\pi}\int\frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot\vec{r}}\, \frac{4\pi}{k'^2}\, i\vec{k}'\times\int\frac{d^3k}{(2\pi)^3}\, i\vec{k}\times\vec{f}(\vec{k})$$
$$\times \underbrace{\int d^3r'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')}$$

do integration over $\vec{k}'$ $\Rightarrow$ $\vec{k}'=\vec{k}$ [illegible]

$$\vec{f}_\perp(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} \; \frac{-\vec{k}\times(\vec{k}\times\vec{f}(k))}{k^2} \; e^{i\vec{k}\cdot\vec{r}}$$

from which we get

$$\vec{f}_\perp(\vec{k}) = \frac{-\vec{k}\times(\vec{k}\times\vec{f}(k))}{k^2} = -\hat{k}\times(\hat{k}\times\vec{f}(k))$$

using the triple product rule for cross products we can also write

$$\vec{f}_\perp(\vec{k}) = -\hat{k}\times(\hat{k}\times f(\vec{k})) = -\hat{k}\,(\hat{k}\cdot\vec{f}(\vec{k})) + \vec{f}(\vec{k})\,(\hat{k}\cdot\hat{k})$$

$$= \vec{f}(k) - \hat{k}\,\hat{k}\cdot\vec{f}(\vec{k}) = \vec{f}(k) - \vec{f}_{\parallel}(k)$$

so $\quad \vec{f}(k) = \vec{f}_{\parallel}(\vec{k}) + \vec{f}_\perp(\vec{k}) \quad$ as it should!

$\vec{f}_\perp(\vec{k})$ is the projection of $\vec{f}(\vec{k})$ onto the plane perpendicular to $\hat{k}$.

Notes

1) The solution above can be worked in reverse to derive the real space relations for $f_\perp$ and $f_{\parallel}$. That is, for any $\vec{f}(\vec{r})$ with a Fourier transform $\vec{f}(\vec{k})$, one can define $\vec{f}_{\parallel}(\vec{k}) \equiv \hat{k}\,\hat{k}\cdot\vec{f}(\vec{k})$ and $\vec{f}_\perp(\vec{k}) \equiv -\hat{k}\times(\hat{k}\times\vec{f}(\vec{k}))$. Taking the Fourier transforms of $\vec{f}_{\parallel}(\vec{k})$ and $\vec{f}_\perp(\vec{k})$ one gets expressions for $\vec{f}_{\parallel}(\vec{r})$ and $\vec{f}_\perp(\vec{r})$ which can be worked into the forms given at the start of the problem. Because $\vec{f}_{\parallel}(\vec{k})$ is $\parallel \vec{k}$, it ~~automati~~ necessarily follows that

$\vec{\nabla} \times \vec{f}_{\parallel}(\vec{r}) = 0$. Because $\vec{f}_{\perp}(\vec{k})$ is $\perp$ $\vec{k}$, it necessarily follows that $\vec{\nabla} \cdot \vec{f}_{\perp}(\vec{r}) = 0$. Thus we have proven that any $\vec{f}(\vec{r})$ can be written as the sum of a curlfree piece and a divergenceless piece, and we have derived the formulas to construct these pieces.

2) From the expressions for $\vec{f}_{\parallel}(\vec{r})$ and $\vec{f}_{\perp}(\vec{r})$ we also see the solution to the general problem of determining a function $\vec{f}(\vec{r})$ if one knows its curl and its divergence. That is, if one knows $\vec{\nabla} \cdot \vec{f}(\vec{r}) = D(\vec{r})$ and $\vec{\nabla} \times \vec{f}(\vec{r}) = \vec{C}(\vec{r})$ with $D$ and $\vec{C}$ known functions, then the solution for $\vec{f}(\vec{r})$ is

$$\vec{f}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3r' \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi} \vec{\nabla} \times \int d^3r' \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$