PHY 415 Solutions HW 2

1) For charge free space (ie $\nabla^2\phi = 0$), the value of ϕ at any point is equal to the average of ϕ over the surface of any sphere centered on that point.

Consider the point at the origin $\vec{r} = 0$ (since we can always shift our coord system to put the origin at any point, this in fact covers the general case)

Consider the volume V of a sphere of surface S and radius R centered at $\vec{r} = 0$.

If $\vec{E} = -\vec{\nabla}\phi$, then $\underbrace{\vec{\nabla}\cdot\vec{E} = 0}_{\nabla^2\phi = 0 \Rightarrow}$, so

$$0 = \int_V d^3r\, \vec{\nabla}\cdot\vec{E} = \oint_S da\, \hat{n}\cdot\vec{E} = \oint_S d\Omega\, R^2\, \hat{r}\cdot\vec{E}$$

$$= -\oint_S d\Omega\, R^2 \frac{\partial\phi}{\partial r}\Big|_{r=R} = -R^2\frac{d}{dr}\oint_S d\Omega\, \phi$$

where $d\Omega \equiv d\theta \sin\theta\, d\phi$ is the differential solid angle. Since

$$\frac{d}{dr}\oint d\Omega\, \phi = 0 \Rightarrow \oint_S d\Omega\, \phi = \text{const} \quad \text{indep of radius of sphere}$$

~~Also $\oint_S d\Omega\,\phi$ is average of ϕ over surface of any sphere~~

To find the value of the const, evaluate over the sphere with radius $R \to 0$. Then

$$\oint_S d\Omega\, \phi(\vec{r}) \simeq \phi(0) \oint_S d\Omega = 4\pi\, \phi(0)$$

since $\phi(\vec{r}) \simeq \phi(0)$ as $R \to 0$

So const $= 4\pi\phi(0)$ and so for any radius R

$$\oint_S d\Omega\, \phi(\vec{r}) = 4\pi\, \phi(0)$$

$$\Rightarrow \boxed{\phi(0) = \frac{1}{4\pi} \oint d\Omega\, \phi(r) = \frac{1}{4\pi R^2} \oint_S d\Omega\, R^2 \phi(r)}$$

= average of ϕ over the surface of the sphere

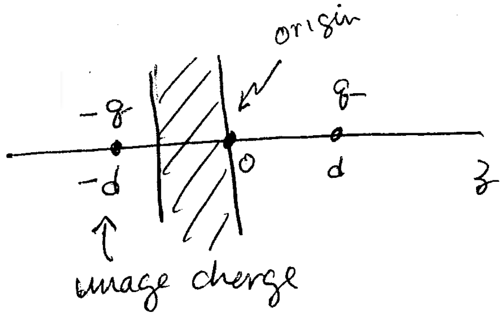
2)

Discussion Question 2.2.1

See:

https://rochester.hosted.panopto.com/Panopto/Pages/Viewer.aspx?id=0515d51a-4662-467c-a7d1-ac3901056718

Discussion Question 2.2.2



We want $\vec{E}$ to be normal to surface on right hand side where q is. The way to do this is to put $-q$ a distance $-d$ behind this surface, exactly as we did for the infinitely thin plane in lecture — the finite width w here does not effect anything!

Now the image charge $-q$ induces a charge $-q$ on the conducting slab. Since the slab has a total net charge Q, we need to add to the slab $Q+q$ in a way that keeps $\vec{E}$ normal to the surface, ~~We can do~~ and keeps $\vec{E}=0$ inside the slab. We can do this by putting $\frac{Q+q}{2}$ uniformly distributed on each surface of the slab

So on the right of the slab

$$\boxed{\vec{E} = \underbrace{q\,\frac{(\vec{r}-d\hat{z})}{|r-d\hat{z}|^3}}_{\text{field from } q} - \underbrace{q\,\frac{(\vec{r}+d\hat{z})}{|\vec{r}+d\hat{z}|^3}}_{\text{field from image } -q} + \underbrace{4\pi\,\frac{(Q+q)}{2A}\,\hat{z}}_{\text{field from unifor } \sigma=\frac{Q+q}{2A} \text{ on right surface}}}$$

Inside the slab

$$\boxed{\vec{E}=0}$$

fields from q and the induced $-q$ from the image cancell, as do the fields from the uniform $\frac{Q+q}{2}$ on each side

On the left side of the slab

fields from q and the induced $-q$ on the right surface cancel. fields from uniform $\frac{Q+q}{2}$ on each surface give

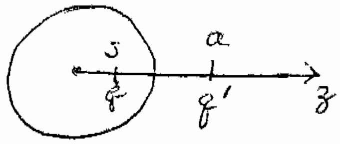
$$\boxed{\vec{E} = -\frac{4\pi(Q+q)}{2A}\hat{z}}$$

Force is force between q and image $-q$ plus force between q and the $\vec{E}$ due to uniform $\frac{Q+q}{2}$ on each surface, ie $\frac{4\pi(Q+q)}{2A}\hat{z}$

$$\boxed{F = \frac{-q^2}{(2d)^2}\hat{z} + \frac{4\pi(Q+q)q}{2A}\hat{z}}$$

3)

In lecture we found that for a charge q, outside a grounded conducting, a distance s from the origin, we could make $\Phi(R,\theta)=0$ by placing an image charge $q' = -qR/s$ inside the sphere, a distance $a = R^2/s$ from the origin. Now, there was nothing in this derivation that specifically required q to be outside – hence the solution works just as well for q inside, only q' now lies at R^2/s outside. (If you want to check this explicitly just make substitutions $q \Leftrightarrow q'$ and $s \Leftrightarrow a$ in the above, ~~regarding the~~ interchanging the roles of charge and image, and you will find the same result!) So for q a dist s inside, the potential is:



$$\phi(\vec{r}) = \frac{q}{|\vec{r}-s\hat{z}|} + \frac{q'}{|\vec{r}-a\hat{z}|}$$

a)

$$\boxed{\phi(r,\theta) = q\left\{\frac{1}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{1}{\left(R^2+\frac{r^2s^2}{R^2}-2rs\cos\theta\right)^{1/2}}\right\}}$$

Note: In solving parts (a) – (c) we are only interested in what happens inside the inner surface, ie $r \le a$. Since we know the charge distribution inside, and we know ϕ on this surface, our solution in NO way depends on anything outside $r=a$, ie does not depend on what happens on outer surface at $r=b$.

b) To get the induced surface charge density we use

$$4\pi\sigma = (\vec{E}^{top} - \vec{E}^{bottom}) \cdot \hat{n}$$

here $\vec{E}^{top} = 0$ field at $r = R^+$ just outside

$\vec{E}^{bottom} = -\frac{d\phi}{dr}\hat{r}$ field at $r = R^-$ just inside

$\hat{n} = \hat{r}$ radial unitvector

$$4\pi\sigma = \left.\frac{d\phi}{dr}\right|_{r=R}$$

Note: although ϕ is the same as with q outside, σ is <u>not</u>! For q outside, $\vec{E}^{top} = -\frac{d\phi}{dr}\hat{r}$ and $\vec{E}^{bottom} = 0$, and $4\pi\sigma = -\left.\frac{d\phi}{dr}\right|_{r=R}$, so there is a <u>sign difference</u>. We will soon see another difference!

$$\sigma = \frac{q}{4\pi}\left\{ \frac{(-\frac{1}{2})(2R - 2s\cos\theta)}{(R^2+s^2-2Rs\cos\theta)^{3/2}} - \frac{(-\frac{1}{2})(2\frac{s^2}{R} - 2Rs\cos\theta)}{(R^2+s^2-2Rs\cos\theta)^{3/2}} \right\}$$

$$\sigma = \frac{q\,(\frac{s^2}{R} - R)}{4\pi(R^2+s^2-2Rs\cos\theta)^{3/2}} = \frac{-q}{4\pi}\frac{R(1-(s/R)^2)}{R^3\left(1+\left(\frac{s}{R}\right)^2 - 2\left(\frac{s}{R}\right)\cos\theta\right)^{3/2}}$$

$$\boxed{\sigma = \frac{-q}{4\pi R^2}\,\frac{1-(s/R)^2}{(1+(s/R)^2 - 2(s/R)\cos\theta)^{3/2}}}$$

Note: as a quick check, notice that as $s \to 0$ (ie charge q is at origin) then $\sigma = \frac{-q}{4\pi R^2}$, a uniform surface charge density of total charge $-q$. We know that this is the correct solution for q at the origin!

The total induced charge is

$$Q_{ind} = 2\pi \int_0^\pi d\theta \sin\theta \, R^2 \, \sigma(\theta)$$

let $\mu = -\cos\theta$
$d\mu = \sin\theta \, d\theta$

$$Q_{ind} = 2\pi R^2 \int_{-1}^{1} d\mu \left(\frac{-q}{4\pi R^2}\right) \frac{1-(s/R)^2}{\left[1+(s/R)^2+2(s/R)\mu\right]^{3/2}}$$

$$= -\frac{q}{2}\left[1-(s/R)^2\right] \left[\frac{2\left(\frac{-R}{2s}\right)}{\left[1+(s/R)^2+2(s/R)\mu\right]^{1/2}}\right]_{-1}^{1}$$

$$= +q\left[1-(s/R)^2\right]\left(\frac{R}{2s}\right)\left[\frac{1}{\left(1+(s/R)^2+2(s/R)\right)^{1/2}} - \frac{1}{\left(1+(s/R)^2-2(s/R)\right)^{1/2}}\right]$$

$$= q\left[1-(s/R)^2\right]\left(\frac{R}{2s}\right)\left[\frac{1}{1+(s/R)} - \frac{1}{1-(s/R)}\right]$$

$$= q\left[1-(s/R)^2\right]\left(\frac{R}{2s}\right) \frac{(1-s/R)-(1+s/R)}{(1+s/R)(1-s/R)}$$

denom $= 1-(s/R)^2$

$$= q\left(\frac{R}{2s}\right)\left(-\frac{2s}{R}\right)$$

$$\boxed{Q_{ind} = -q}$$

Note: Q_{ind} is <u>not</u> equal to the image charge.
Q_{ind} is negative of the charge inside the sphere $-q$.

c) The force acting on q is due to the electric field of the image charge. The magnitude of the force is

$$F = \frac{|q q'|}{(s-a)^2} = \frac{q^2 (R/s)}{(s - R^2/s)^2}$$

$$\boxed{F = q^2 \frac{s}{R^3} \frac{1}{[1-(s/R)^2]^2}}$$

Note: as $s \to 0$, $F \to 0$ as it should

Since q and q' are oppositely charged, the force is attractive. Hence $\vec{F}$ is in the $+\hat{z}$ direction, q is attracted towards the spherical wall

$\Rightarrow$ $s=0$ is a point of unstable mechanical equilibrium. Any small displacement from the origin and the charge q will move to the wall at $r=R$.

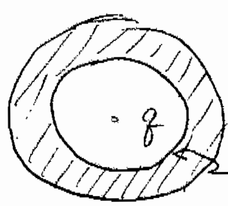
For $s = R-d$ near the wall, ie $d \ll R$,

$$F = q^2 \frac{(R-d)}{R^3} \frac{1}{[1-(1-\frac{d}{R})^2]^2} \simeq \frac{q^2}{R^2} \frac{1}{[1-(1-\frac{2d}{R}+\frac{d^2}{R^2})]}$$

$$\simeq \frac{q^2}{R^2} \frac{R^2}{4d^2} = \frac{q^2}{4d^2}$$

same as for infinite flat plane

d) If the sphere has a fixed charge Q on it, then we need to add additional charge $Q+q$ to it in such a way that the total $\vec{E}$ field stays normal to the surface. The way to do this is to put the $Q+q$ charge uniformly spread out on the <u>outside</u> surface on the sphere

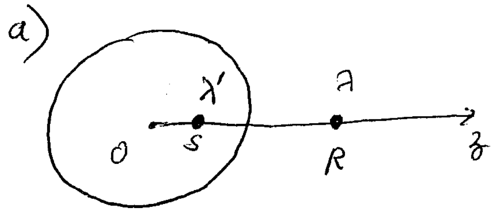


Distributing the charge in this way creates <u>no</u> additional $\vec{E}$ fields in the interior – it has no effect on the force on q or on the induced σ on the inner surface. All it does is add a constant to the potential.

Similarly, if the sphere is held at some fixed potential $\Phi_0 \neq 0$, this induces some charge Q onto the sphere, which gets distributed just like above and results in no changes in the force on q or the induced σ on the inner surface.

4)

a)



try to place image line charge λ' at distance s from origin as shown.

For a line charge λ at the origin, we had

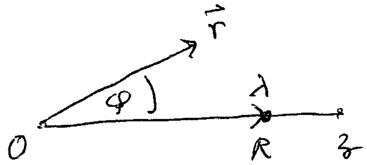
$$\phi(r) = -2\lambda \ln r + C$$

where r is the cylindrical radial coord in the xy plane, and C is a constant

Generalizing to a line charge at position $\vec{r}_0$ in xy plane gives

$$\phi(\vec{r}) = -2\lambda \ln|\vec{r}-\vec{r}_0| + C = -\lambda \ln(\vec{r}-\vec{r}_0)^2 + C$$

For (r, φ) the usual cylindrical coords, we have



for the configuration of line charge and image line charge above:

$$\phi(\vec{r}) = -\lambda \ln\left[r^2 + R^2 - 2rR\cos\varphi\right] + C_1$$
$$\qquad - \lambda' \ln\left[r^2 + s^2 - 2rs\cos\varphi\right] + C_2$$

we have to choose λ' and s so that on the surface of the cylinder at $r=b$, ϕ is a constant ϕ_0

$$\phi(b,\varphi) = -\lambda \ln\left[b^2+R^2-2bR\cos\varphi\right] + C_1$$
$$-\lambda' \ln\left[b^2+s^2-2bs\cos\varphi\right] + C_2 = \phi_0$$

Proceeding as we did for the image charge inside the sphere, we see that we can make the terms involving the angular dependence φ look similar by choosing s as follows:

$$b^2+s^2-2bs\cos\varphi = \left(\frac{s}{R}\right)\left(\frac{b^2R}{s} + \frac{s^2R}{s} - 2bR\cos\varphi\right)$$

$$= \left(\frac{s}{R}\right)\left(\frac{b^2R}{s} + sR - 2bR\cos\varphi\right)$$

Choose $sR = b^2 \rightarrow \boxed{s = b^2/R}$ to get

$$= \left(\frac{s}{R}\right)\left(R^2+b^2-2bR\cos\varphi\right)$$

$$= \left(\frac{b}{R}\right)^2\left(R^2+b^2-2bR\cos\varphi\right)$$

So then

$$\phi(b,\varphi) = -\lambda \ln\left[b^2+R^2-2bR\cos\varphi\right] + C_1$$
$$-\lambda' \ln\left(\frac{b}{R}\right)^2 - \lambda' \ln\left[R^2+b^2-2bR\cos\varphi\right] + C_2$$

the φ dependence will cancel if we take $\boxed{\lambda' = -\lambda}$,

- unlike the spherical case, λ' does <u>not</u> depend on position of line charge R!

So $\phi(r,\varphi) = -\lambda \ln\left[r^2+R^2-2rR\cos\varphi\right] + C_1$
$$+\lambda \ln\left[r^2+\frac{b^4}{R^2}-2r\frac{b^2}{R}\cos\varphi\right] + C_2$$

as $r\to\infty$ the above gives

$$\phi(r\to\infty,\varphi) \simeq -\lambda\ln r^2 + \lambda \ln r^2 + C_1 + C_2$$
$$= C_1 + C_2$$

which we want, by the given boundary condition, to vanish.
So $C_1+C_2=0$

So

b) $$\boxed{\begin{aligned}\phi(r,\varphi) &= -\lambda\ln\left[r^2+R^2-2rR\cos\varphi\right]\\ &\quad+\lambda\ln\left[r^2+\frac{b^4}{R^2}-2r\frac{b^2}{R}\cos\varphi\right]\end{aligned}}$$

on the surface of the cylinder:

$$\phi_0 = \phi(b,\varphi) = -\lambda\ln\left[b^2+R^2-2bR\cos\varphi\right]$$
$$+\lambda\ln\left[b^2+\frac{b^4}{R^2}-\frac{2b^3}{R}\cos\varphi\right]$$
$$= -\lambda\ln\left[b^2+R^2-2bR\cos\varphi\right]$$
$$+\lambda\ln\left(\frac{b}{R}\right)^2 + \lambda\ln\left[b^2+R^2-2bR\cos\varphi\right]$$

$$\boxed{\phi_0 = 2\lambda\ln(b/R)}$$

c) surface charge density given by

$$4\pi\sigma(\varphi) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=b}$$

from part (b)

$$-\frac{\partial\phi}{\partial r} = \frac{+\lambda\,(2r - 2R\cos\varphi)}{r^2 + R^2 - 2rR\cos\varphi} - \frac{\lambda\,(2r - \frac{2b^2}{R}\cos\varphi)}{r^2 + \frac{b^4}{R^2} - \frac{2rb^2}{R}\cos\varphi}$$

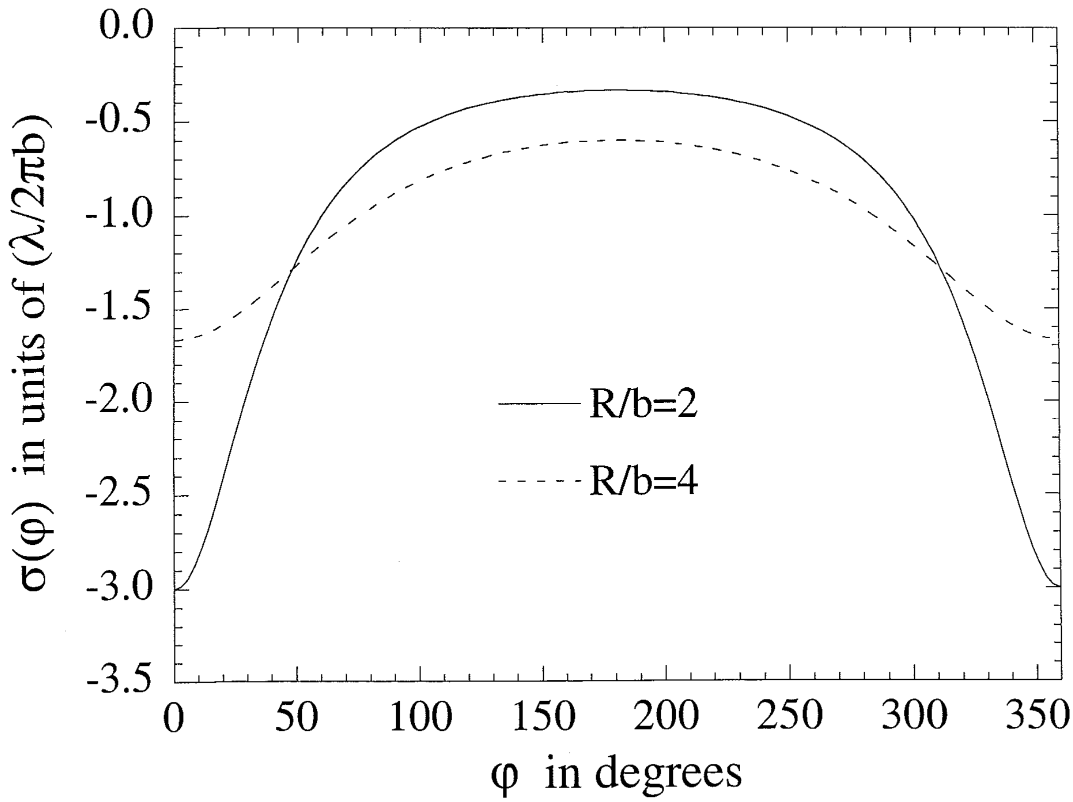
evaluate above at $r = b$ to get

$$4\pi\sigma(\varphi) = \frac{\lambda\,(2b - 2R\cos\varphi)}{b^2 + R^2 - 2bR\cos\varphi} - \frac{\lambda\,(2b - \frac{2b^2}{R}\cos\varphi)}{b^2 + \frac{b^4}{R^2} - \frac{2b^3}{R}\cos\varphi}$$

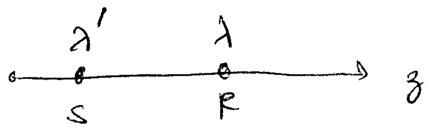
$$= \frac{\lambda(2b - 2R\cos\varphi)}{b^2 + R^2 - 2bR\cos\varphi} - \frac{\lambda\left(\frac{R^2}{b^2}\right)\left(2b - \frac{2b^2}{R}\cos\varphi\right)}{R^2 + b^2 - 2bR\cos\varphi}$$

$$4\pi\sigma(\varphi) = \frac{2\lambda\left(b - \frac{R^2}{b}\right)}{b^2 + R^2 - 2bR\cos\varphi}$$

$$\boxed{\sigma(\varphi) = \frac{\lambda}{2\pi b}\;\frac{1 - (R/b)^2}{1 + (R/b)^2 - 2(R/b)\cos\varphi}}$$


0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
σ(φ) in units of (λ/2πb)
R/b=2
R/b=4
0
50
100
150
200
250
300
350
φ in degrees

d) The force on the line charge λ per unit length is just the force from the image λ'



The $\vec{E}$ field from λ' is $\quad \vec{E} = \dfrac{2\lambda' \, (\vec{r} - s\hat{z})}{|\vec{r} - s\hat{z}|^2}$

The force on λ is then

$$\vec{F} = \lambda \vec{E} = \frac{2\lambda\lambda' \, (R\hat{z} - s\hat{z})}{(R\hat{z} - s\hat{z})^2}$$

use $\lambda' = -\lambda$

$$\boxed{\vec{F} = \frac{-2\lambda^2}{R-s}\,\hat{z}}$$

force is attractive

where $s = b^2/R$

$$\vec{F} = \frac{-2\lambda^2}{R - b^2/R} = \frac{-2\lambda^2}{R}\,\frac{1}{1-(b/R)^2}$$

Note, as R gets close to the cylinder, ie $R = b+d$ for $d \ll b$, then $\vec{F} \cong \dfrac{-2\lambda^2}{b+d}\,\dfrac{1}{1-\left(\frac{b}{b+d}\right)^2}$

$$\cong \frac{-2\lambda^2}{b}\,\frac{1}{1-\left(1-\frac{2d}{b}\right)} = \frac{-\lambda^2}{d}$$

This is same result as for a line charge in front of an infinite flat plane!