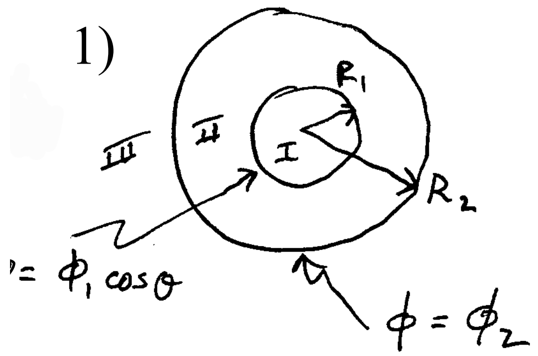
Solutions Problem Set 3

1)



$\phi \to 0$ as $r \to \infty$

azimuthal symmetry
$\to$ use Legendre polynomials

a) Region I, $r < R_1$

$$\phi(r) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

no $\frac{B_\ell}{r^{\ell+1}}$ terms since ϕ finite as $r \to 0$

at $r = R_1$

$$\phi(R_1) = \phi_1 \cos\theta = \sum_\ell A_\ell R_1^\ell P_\ell(\cos\theta)$$

since $P_1(\cos\theta) = \cos\theta \Rightarrow A_1 R_1 = \phi_1 \Rightarrow A_1 = \frac{\phi_1}{R_1}$

all other $A_\ell = 0$

$$\boxed{\phi(r) = \phi_1 \frac{r}{R_1} \cos\theta \qquad r < R_1}$$

b) Region III, $r > R_2$

$$\phi(r) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

no $A_\ell r^\ell$ terms since $\phi \to 0$ as $r \to \infty$

at $r = R_2$

$$\phi_2 = \phi(R_2) = \sum_\ell \frac{B_\ell}{R_2^{\ell+1}} P_\ell(\cos\theta)$$

since $P_0(\cos\theta) = 1 \Rightarrow \frac{B_0}{R_2} = \phi_2 \Rightarrow B_0 = \phi_2 R_2$

all other $B_\ell = 0$

$$\boxed{\phi(r) = \Phi_2 \frac{R_2}{r} \qquad r > R_2}$$

c) Region II $\quad R_1 < r < R_2$

$$\phi(r) = \sum_\ell \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$$

here both A_ℓ and B_ℓ terms are needed.

ϕ continuous at R_1 and R_2 so match onto solutions in part (a) to find A_ℓ and B_ℓ

i) $$\phi(R_2) = \sum_\ell \left[A_\ell R_2^\ell + \frac{B_\ell}{R_2^{\ell+1}}\right] P_\ell(\cos\theta) = \Phi_2$$

$$\Rightarrow \quad A_0 + \frac{B_0}{R_2} = \phi_2 \quad \text{and} \quad A_\ell R_2^\ell = -\frac{B_\ell}{R_2^{\ell+1}} \quad \text{all other } \ell$$

$$A_\ell = \frac{-B_\ell}{R_2^{2\ell+1}} \quad \ell = 1, 2, 3, \ldots$$

ii) $$\phi(R_1) = \sum_\ell \left[A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}}\right] P_\ell(\cos\theta) = \Phi_1 \cos\theta$$

$$\Rightarrow \quad A_1 R_1 + \frac{B_1}{R_1^2} = \phi_1 \quad \text{and} \quad A_\ell = -\frac{B_\ell}{R_1^{2\ell+1}} \quad \text{all other } \ell$$

$$\ell = 0, 2, 3, \ldots$$

consider $\ell \neq 0, 1$

$$\left.\begin{aligned} &\text{from (i)} \quad A_\ell = -\frac{B_\ell}{R_2^{2\ell+1}} \\ &\text{from (ii)} \quad A_\ell = \frac{-B_\ell}{R_1^{2\ell+1}} \end{aligned}\right\}$$

can't both be satisfied (since $R_1 \neq R_2$) unless $A_\ell = B_\ell = 0$

for $l=0$ from (i) $A_0 + \frac{B_0}{R_2} = \phi_2$
from (ii) $A_0 + \frac{B_0}{R_1} = 0$
} solve for A_0, B_0

for $l=1$ from (i) $A_1 + \frac{B_1}{R_2^3} = 0$
from (ii) $A_1 R_1 + \frac{B_1}{R_1^2} = \phi_1$
} solve for A_1, B_1

For $l=0$, $A_0 = -\frac{B_0}{R_1} \Rightarrow B_0\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = \phi_2$

$$B_0 = \frac{\phi_2 R_1 R_2}{R_1 - R_2}$$

$$A_0 = -\frac{\phi_2 R_2}{R_1 - R_2}$$

For $l=1$, $A_1 = \frac{-B_1}{R_2^3} \Rightarrow B_1\left(\frac{1}{R_1^2} - \frac{R_1}{R_2^3}\right) = \phi_1$

$$B_1 = \frac{\phi_1 R_1^2 R_2^3}{R_2^3 - R_1^3}$$

$$A_1 = -\frac{\phi_1 R_1^2}{R_2^3 - R_1^3}$$

$$\boxed{\phi(r) = \frac{\phi_2 R_2}{R_2 - R_1}\left[1 - \frac{R_1}{r}\right] - \frac{\phi_1 R_1^2}{R_2^3 - R_1^3}\left[r - \frac{R_2^3}{r^2}\right]\cos\theta}$$

$R_1 < r_1 < R_2$

d) Surface charge

$$4\pi\sigma = -\frac{\partial\phi^{out}}{\partial r} + \frac{\partial\phi^{in}}{\partial r}$$

at R_1, ϕ^{in} is ϕ in region I
ϕ^{out} is ϕ in region II

$$\left.\frac{\partial\phi^{in}}{\partial r}\right|_{R_1} = \frac{\phi_1}{R_1}\cos\theta$$

$$\left.\frac{\partial\phi^{out}}{\partial r}\right|_{R_1} = \frac{\phi_2 R_2}{R_2-R_1}\frac{R_1}{R_1^2} - \frac{\phi_1 R_1^2}{R_2^3-R_1^3}\left[1+\frac{2R_2^3}{R_1^3}\right]\cos\theta$$

$$4\pi\sigma = \frac{-\phi_2 R_2}{(R_2-R_1)R_1} + \left[\frac{\phi_1 R_1^2}{R_2^3-R_1^3}\left[1+\frac{2R_2^3}{R_1^3}\right] + \frac{\phi_1}{R_1}\right]\cos\theta$$

$$\boxed{4\pi\sigma = \frac{-\phi_2 R_2}{(R_2-R_1)R_1} + \phi_1\left[\frac{1}{R_1} + \frac{R_1^2}{R_2^3-R_1^3}\left(1+\frac{2R_2^3}{R_1^3}\right)\right]\cos\theta \quad \text{at } r=R_1}$$

at R_2, ϕ^{in} is ϕ in region II
ϕ^{out} is ϕ in region III

$$\left.\frac{\partial\phi^{in}}{\partial r}\right|_{R_2} = \frac{\phi_2 R_2}{R_2-R_1}\frac{R_1}{R_2^2} - \frac{\phi_1 R_1^2}{R_2^3-R_1^3}\left[1+\frac{2R_2^3}{R_2^3}\right]\cos\theta$$

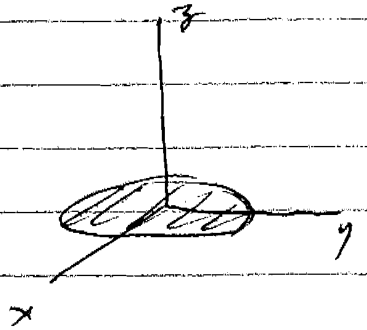
$$\left.\frac{\partial\phi^{out}}{\partial r}\right|_{R_2} = \frac{-\phi_2 R_2}{R_2^2} = -\frac{\phi_2}{R_2}$$

$$\boxed{\begin{gathered} 4\pi\sigma = \frac{\Phi_2}{R_2} + \frac{\Phi_2 R_1}{R_2(R_2-R_1)} - \frac{3\phi_1 R_1^2}{R_2^3-R_1^3}\cos\theta \\ r = R_2 \end{gathered}}$$

2)

Uniformly charged disk of radius R in xy plane

a) $\phi(\vec{r}) = \int da' \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|}$ Coulomb formula



apply to present case

$\sigma(\vec{r}') = \sigma_0$ constant

$da' = d\phi\, dr' r'$ $\quad 0 \le r' \le R$

$0 \le \phi < 2\pi$

$\vec{r}' = r'\hat{r}$

$\vec{r} = z\hat{z}$ on $\hat{z}$ axis

$$\phi(z\hat{z}) = \int_0^{2\pi} d\varphi \int_0^R dr' \frac{r'\sigma_0}{|z\hat{z} - r'\hat{r}|}$$

$$= 2\pi \int_0^R dr' \frac{r'\sigma_0}{\sqrt{z^2+r'^2}}$$ since $z\hat{z}$ is $\perp$ $r'\hat{r}$

$$= 2\pi\sigma_0 \left[(z^2+r'^2)^{1/2} \right]_0^R$$

$$\boxed{\phi(z\hat{z}) = 2\pi\sigma_0 \left[\sqrt{z^2+R^2} - |z| \right]}$$

potential along z axis

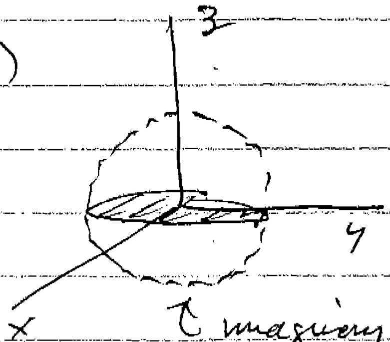
Note the $|z|$ in the result —

this is needed to take care of $z<0$

since $\sqrt{z^2} = |z| = -z$ for $z<0$

need $\phi(z) = \phi(-z)$ symmetric

b)



↖ imaginary sphere at radius R

azimuthal symmetry ⇒ can express $\phi(\vec{r})$ in terms of expansion in Legendre polynomials.

General form of solution is

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

① Solve first for $r > R$.

$\Rightarrow A_\ell = 0$ for $r > R$ since we want $\phi \to 0$ as $r \to \infty$. Only the B_ℓ do not vanish

To solve for the B_ℓ we use ~~for~~ the boundary condition the exact solution on the z axis, computed in part (a). Along the positive z axis:

$$\phi(r, \theta = 0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} \qquad \text{since } P_\ell(1) = 1$$

$$= 2\pi\sigma_0 \left[\sqrt{r^2 + R^2} - r \right]$$ from part (a), using $z = r$

we want to expand above in Taylor series

$$\sqrt{r^2+R^2} - r = r\left[\sqrt{1 + \left(\frac{R}{r}\right)^2} - 1 \right]$$

$$= r\left[\frac{1}{2}\left(\frac{R}{r}\right)^2 - \frac{1}{8}\left(\frac{R}{r}\right)^4 + \frac{1}{16}\left(\frac{R}{r}\right)^6 - \cdots \right.$$

$$= \frac{1}{2}\frac{R^2}{r} - \frac{1}{8}\frac{R^4}{r^3} + \frac{1}{16}\frac{R^6}{r^5} - \cdots$$

So $\frac{B_0}{r} + \frac{B_1}{r^2} + \frac{B_2}{r^3} + \frac{B_3}{r^4} + \frac{B_4}{r^5} + \frac{B_5}{r^6} + \ldots$

$$= 2\pi\sigma_0 \left[\frac{1}{2}\frac{R^2}{r} - \frac{1}{8}\frac{R^4}{r^3} + \frac{1}{16}\frac{R^6}{r^5} - \ldots \right]$$

next term is $\sim \frac{1}{r^7}$

$$\Rightarrow \boxed{\begin{aligned} B_0 &= \pi R^2 \sigma_0 \\ B_1 &= 0 \\ B_2 &= -\frac{1}{4}\pi R^4 \sigma_0 \\ B_3 &= 0 \\ B_4 &= \frac{1}{8}\pi R^6 \sigma_0 \\ B_5 &= 0 \end{aligned}}$$

all the B_ℓ for ℓ odd vanish

One can verify that the above expansion holds both above and below the charged disk for $r \geq R$

On physical grounds, since one can go from above ($\theta < \frac{\pi}{2}$) to below ($\theta > \pi/2$) for $r \geq R$ without crossing any charged surfaces, the solution should be smooth between the two regions

On mathematical grounds, we could always repeat the above calculation, but along the $-z$ axis, $\theta = \pi$. Here the Taylor expansion of the exact solution remains the same. The Legendre expansion becomes $\sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\pi) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} (-1)^\ell$

where we used $P_\ell(-1) = (-1)^\ell$. So the odd terms in the expansion change sign for $\theta = \pi$. But we saw that all the B_ℓ for odd ℓ vanish! so the solutions we get for the even ℓ B_ℓ are the same as at $\theta = 0$.

we therefore get for $\underline{r > R \text{, all } \theta}$

$$\phi(r,\theta) = \frac{B_0}{r} P_0(\cos\theta) + \frac{B_2}{r^3} P_2(\cos\theta) + \frac{B_4}{r^5} P_4(\cos\theta$$

$$= \frac{q}{r} - \frac{qR^2}{4r^3}\left[\tfrac{1}{2}(3\cos^2\theta - 1)\right] + \frac{qR^4}{8r^5} P_4(\cos\theta)$$

where $q \equiv \pi R^2 \sigma_0$ is total charge on the disk

$$\phi(r,\theta) = \frac{q}{r}\left\{1 - \frac{R^2}{4r^2} P_2(\cos\theta) + \frac{R^4}{8r^4} P_4(\cos\theta)\right\}$$

expansion in $\left(\frac{R}{r}\right)^2$

(2) Solve for $r < R$ above the disk $\theta < \pi/2$

Here we must have all $B_\ell = 0$ since ϕ should not diverge as $r \to 0$

Solve for the A_ℓ by matching exact solution on positive z axis.

$$\phi(r, \theta=0) = \sum_{\ell=0}^{\infty} A_\ell r^\ell = 2\pi\sigma_0 \left[\sqrt{r^2+R^2} - r\right]$$

Now expand

$$\sqrt{r^2+R^2} - r = R\sqrt{1+\left(\frac{r}{R}\right)^2} - r$$

$$= R\left\{1 + \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^4 + \frac{1}{16}\left(\frac{r}{R}\right)^6 - \cdots\right.$$

$$= R - r + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \frac{1}{16}\frac{r^6}{R^5} - \cdots$$

So

$$A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + A_5 r^5 + \cdots$$

$$= 2\pi\sigma_0 \left\{ R - r + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \frac{1}{16}\frac{r^6}{R^5} \right\}$$

$\theta < \frac{\pi}{2}$ above plane

$$\boxed{\begin{array}{ll} A_0 = 2\pi R\sigma_0 & A_3 = 0 \\ A_1 = -2\pi\sigma_0 & A_4 = -\dfrac{\pi\sigma_0}{4R^3} \\ A_2 = +\dfrac{\pi\sigma_0}{R} & A_5 = 0 \\ & A_6 = \dfrac{\pi\sigma_0}{8R^5} \end{array}}$$

$A_\ell = 0$ for all ℓ odd except A_1

$$\phi(r,\theta) = A_0 + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_4 r^4 P_4(\cos\theta) + \cdots$$

$$= \frac{2q}{R} - \frac{2qr}{R^2} P_1(\cos\theta) + \frac{q r^2}{R^3} P_2(\cos\theta) - \frac{q r^4}{4R^5} P_4(\cos\theta);$$

So for $r<R$, $\theta<\frac{\pi}{2}$

$$\phi(r,\theta) = \frac{q}{R}\left\{2 - 2\left(\frac{r}{R}\right)P_1(\cos\theta) + \left(\frac{r}{R}\right)^2 P_2(\cos\theta) - \frac{1}{4}\left(\frac{r}{R}\right)^4 P_4(\cos\theta) + \cdots\right.$$

expansion in $\left(\frac{r}{R}\right)$

③ Solve for $r<R$ below the disk $\theta>\pi/2$
match exact solution along negative z axis

$$\phi(r,\theta=\pi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\pi)) = \sum_l A_l r^l (-1)^l$$

$$= 2\pi\sigma_0\left[\sqrt{r^2+R^2} - r\right]$$

$$= 2\pi\sigma_0\left\{R - r + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \frac{1}{16}\frac{r^6}{R^5} - \cdots\right\}$$

the right hand side, from the exact solution, is the same as before. The left hand side changes sign for all the l odd terms. Since all $A_l = 0$ for l odd except $l=0$, all the A_l are the same as they were above the plane, except for A_1 which changes sign

$\theta>\frac{\pi}{2}$ below plane

$$\boxed{\begin{array}{lll} A_0 = 2\pi R\sigma_0 & A_3 = 0 & A_6 = \frac{\pi\sigma_0}{8R^5} \\ A_1 = +2\pi\sigma_0 & A_4 = \frac{-\pi\sigma_0}{4R^3} & \\ A_2 = \frac{\pi\sigma_0}{R} & A_5 = 0 & \end{array}}$$

Note: the solution for $r < R$ above the plane is not the same as the solution below the plane (A_1 changes sign). This is because we can not go from above to below without crossing the charged surface

Note: Although Φ must be continuous at $r = R$, the B_ℓ for $r > R$, and the A_ℓ for $r < R$ do not obey the relation $B_\ell = A_\ell R^{2\ell+1}$ which we derived earlier when we considered continuity of Φ at a conducting shell. How can this be? The series for Φ for $r > R$ is an expansion in powers of $\left(\frac{R}{r}\right)$. The series for Φ for $r < R$ is an expansion in powers of $\left(\frac{r}{R}\right)$. As $r \to R$, these series are poorly convergent.

c) Electric field

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial \phi}{\partial r}\,\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\,\hat{\theta}$$

above

$$\vec{E}_{above} = \frac{q}{R}\left\{ \frac{2}{R} P_1(\cos\theta) - \frac{2r}{R^2} P_2(\cos\theta) + \frac{r^3}{R^4} P_4(\cos\theta) \right\}\hat{r}$$

$$+ \frac{q}{R}\left\{ \frac{2}{R}\frac{\partial P_1(\cos\theta)}{\partial\theta} - \frac{r}{R^2}\frac{\partial P_2(\cos\theta)}{\partial\theta} + \frac{r^3}{R^4}\frac{\partial P_4(\cos\theta)}{\partial\theta} \right\}\hat{\theta}$$

below

$$\vec{E}_{below} = \frac{q}{R}\left\{ -\frac{2}{R} P_1(\cos\theta) - \frac{2r}{R^2} P_2(\cos\theta) + \frac{r^3}{R^4} P_4(\cos\theta) \right\}\hat{r}$$

$$+ \frac{q}{R}\left\{ -\frac{2}{R}\frac{\partial P_1(\cos\theta)}{\partial\theta} - \frac{r}{R^2}\frac{\partial P_2(\cos\theta)}{\partial\theta} + \frac{r^3}{R^4}\frac{\partial P_4(\cos\theta)}{\partial\theta} \right\}\hat{\theta}$$

↑ same as for $\vec{E}_{above}$, except A_1 changes sign

$$\Rightarrow \vec{E}_{above} - \vec{E}_{below} = \frac{q}{R}\left\{ \frac{4}{R} P_1(\cos\theta)\,\hat{r} + \frac{4}{R}\frac{\partial P_1(\cos\theta)}{\partial\theta}\,\hat{\theta} \right\}$$

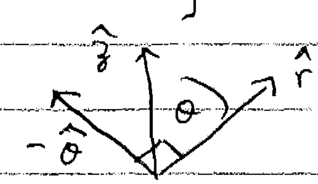
Now $P_1(\cos\theta) = \cos\theta$

$$\frac{\partial P_1(\cos\theta)}{\partial\theta} = -\sin\theta$$

$$\vec{E}_{above} - \vec{E}_{below} = \frac{4q}{R^2}\left\{ \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \right\}$$

use $q = \pi R^2 \sigma_0$

$\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} = \hat{z}$



$$\boxed{\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma_0\,\hat{z}}$$ as it must be at a charged surface

Note: we can write for $\vec{E}_{above}$

$$\vec{E}_{above} = \frac{2q}{R^2}\left\{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right\} \quad \leftarrow \text{leading terms } \ell=1$$

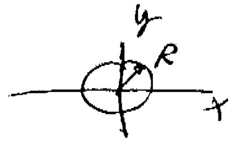
$$+\frac{q}{R^2}\left\{\left[-\frac{2r}{R}P_2(\cos\theta) + \frac{r^3}{R^3}P_4(\cos\theta)\right]\hat{r}\right.$$

$$\left. + \left[-\frac{r}{R}\frac{\partial P_2(\cos\theta)}{\partial\theta} + \frac{r^3}{R^3}\frac{\partial P_4(\cos\theta)}{\partial\theta}\right]\hat{\theta}\right\}$$

leading term is just $2\pi\sigma_0\,\hat{z}$ $(\ell=1)$

this is just what one would expect for an infinite flat plane.

The higher order $\ell=2$, $\ell=4$ etc terms are the edge effect corrections. The leading correction goes proportional to (r/R). So near the center of the disk they are small; they grow large as one approaches the edge $r=R$.

3) Disk in xy plane at $z=0$  uniform σ

use $\rho(\vec{r}) = \sigma(x,y)\,\delta(z)$ $\qquad$ $\delta(z)$ confines to $z=0$ plane

monopole

$$q = \int d^3r\, \rho(\vec{r}) = \int dx\,dy\, \sigma(x,y) = \boxed{\sigma \pi R^2 = q}$$

dipole

$$\vec{p} = \int d^3r\, \rho(\vec{r})\, \vec{r} = \int dx\,dy\, \sigma(x,y) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$\boxed{\vec{p} = 0}$ by symmetry - integrand is antisymmetric in $x \to -x$ and $y \to -y$

quadrupole

$$Q_{ij} = \int d^3r\, \rho(\vec{r}) \left[3 r_i r_j - r^2 \delta_{ij} \right]$$

$$= \int dx\,dy\, \sigma(x,y) \begin{bmatrix} 3x^2 - (x^2+y^2) & 3xy & 0 \\ 3xy & 3y^2 - (x^2+y^2) & 0 \\ 0 & 0 & -(x^2+y^2) \end{bmatrix}$$

where zero's come because we are confined to plane $z=0$ so all terms like z^2, xz, yz vanish.

$$Q_{ij} = \int dx\,dy\, \sigma(x,y) \begin{bmatrix} 2x^2 - y^2 & 3xy & 0 \\ 3xy & 2y^2 - x^2 & 0 \\ 0 & 0 & -(x^2+y^2) \end{bmatrix}$$

Consider $\int dx \int dy\, \sigma(x,y)\, xy = 0$ by symmetry since integrand is antisymmetric under $x \to -x$, and $y \to -y$ separately

$$\int dx \int dy\, \sigma(x,y)\, x^2 = \int dx \int dy\, \sigma(x,y)\, y^2 \quad \text{by rotation symmetry}$$

$$= \frac{1}{2} \int dx\,dy\, \sigma(x,y)\, [x^2+y^2]$$

$$= \frac{1}{2} \int_0^R dr\, r \int_0^{2\pi} d\phi\, \sigma\, r^2 = \pi\sigma \int_0^R dr\, r^3 = \frac{\pi\sigma R^4}{4}$$

$$Q_{ij} = \frac{\pi\sigma R^4}{4} \begin{bmatrix} 2-1 & 0 & 0 \\ 0 & 2-1 & 0 \\ 0 & 0 & -(1+1) \end{bmatrix} = \boxed{\frac{\pi\sigma R^4}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}} \text{ " } Q_{ij}$$

<u>potential</u>

$$\frac{1}{2}\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r} = \frac{\pi\sigma R^4}{8}\, \hat{r}\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$= \frac{\pi\sigma R^4}{8} (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\theta \\ -2\cos\theta \end{pmatrix}$$

$$= \frac{\pi\sigma R^4}{8} \left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi - 2\cos^2\theta \right)$$

$$= \frac{\pi\sigma R^4}{8} \left(\sin^2\theta - 2\cos^2\theta \right) = \frac{\pi\sigma R^4}{8} (1 - 3\cos^2\theta)$$

$$\phi = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{2}\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3}$$

$$\phi = \frac{\sigma \pi R^2}{r} + \frac{\pi \sigma R^4}{8 r^3}(1-3\cos^2\theta)$$

$$\boxed{\phi = \frac{\pi \sigma R^2}{r} + \frac{\pi \sigma R^4}{8 r^3}(1-3\cos^2\theta)}$$

Noting that $P_0(\cos\theta) = 1$ and
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

we can write

$$\phi = \frac{\pi \sigma R^2}{r} P_0(\cos\theta) - \frac{\pi \sigma R^4}{4 r^3} P_2(\cos\theta)$$

In this form we see that our answer exactly agrees with the solution to ~~[illegible]~~ part (b), as obtained there by the separation of variables method