# PHYS 415 Solutions Problem Set 5

1) Center of mass or center of charge?

When we wrote for the contribution of the molecules to the average charge density

$$\sum_n \langle \rho_n \rangle = \left\langle \sum_n q_n \delta(\vec{r}-\vec{r}_n) \right\rangle - \vec{\nabla}\cdot\vec{P} \quad \text{with } \vec{P} = \left\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \right\rangle$$

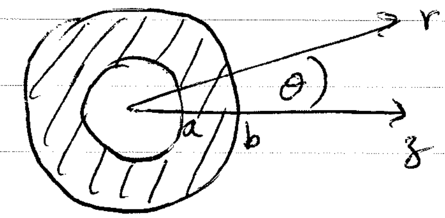
The question arises what should one take for $\vec{r}_n$, the coordinate that defines where molecule $n$ is located. If the molecules are charged, $q_n \neq 0$, then we know that we can choose $\vec{r}_n$ to be the center of charge of the molecule $\sum_{i\in n} q_i \vec{r}_i / \sum_i q_i$ and then $\vec{p}_n = 0$ when computed about that origin, and so $\vec{P} = 0$ and there would be no polarization density. This might sound like it would be the "best choice". [when $q_n = 0$, then $\vec{p}_n$ is independent of what is chosen for $\vec{r}_n$]

However, if one used $\vec{r}_n$ as the center of charge, then one would have to know how $\vec{r}_n$ moved in response to $\vec{E}$ fields or other forces that act on the molecule in order to correctly compute the contribution of the first term $\langle \sum_n q_n \delta(\vec{r}-\vec{r}_n) \rangle$ to the average charge density. Doing that ~~generally often not~~ generally not the simplest thing to do. If one is talking about polarization in a crystal, the positions of the molecules (or atoms) should generally be considered as stationary, while the center of charge is not — heavy neuclei stay still but electrons can shift in applied $\vec{E}$. So if one used $\vec{r}_n$ as the center of charge, one would need

to consider this shift when computing the first term.
But if one took $\vec{r}_n$ as the center of mass, then the first term does not change (since the heavy ~~nuclei~~ nuclei do not move). The $\vec{p}_n$ about the center of mass then becomes finite and the change in average charge density will come from the second $-\vec{\nabla}\cdot\vec{P}$ term rather than from the first $\langle \sum_n q_n \delta(\vec{r}-\vec{r}_n)\rangle$ term. It is generally easier to do it this way - from both conceptual and calculational point of view.

Similarly, if one is talking about the polarization of a charged molecular gas, where molecules are randomly moving about, then too it is the center of mass not the center of charge for which it is easiest to compute the molecule's motion. Thus again it is more convenient to take $\vec{r}_n$ as the center of mass, and use a finite $\vec{p}_n$ when computed about the center of mass.

2)




Region I: $r<a$

II: $a<r<b$

III: $b<r$

$\longrightarrow \vec{E}_0 = E_0 \hat{z}$ azimuthal symmetry

$$\Phi_I(\vec{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$ (no $\frac{1}{r^{l+1}}$ terms as $\phi$ finite at $r=0$)

$$\Phi_{II}(\vec{r}) = \sum_{l=0}^{\infty} \left[ A_l' r^l P_l(\cos\theta) + \frac{B_l'}{r^{l+1}} P_l(\cos\theta) \right]$$

$$\Phi_{III}(\vec{r}) = \underbrace{-E_0 r\cos\theta}_{\text{gives applied field}} + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$ (no $r^l$ terms as deviation from uniform field should vanish as $r \to \infty$)

Boundary conditions

Tangential components $\vec{E}$ continuous $\Rightarrow$ $\phi$ continuous

$\Rightarrow$ (1) $\Phi_I(a) = \Phi_{II}(a)$

(2) $\Phi_{II}(b) = \Phi_{III}(b)$

Normal components $\vec{D}$ continuous (as no free surface charge)

$\rightarrow$ (3) $\dfrac{\partial \Phi_I}{\partial r}(a) = \epsilon \dfrac{\partial \Phi_{II}}{\partial r}(a)$

(4) $\epsilon \dfrac{\partial \Phi_{II}}{\partial r}(b) = \dfrac{\partial \Phi_{III}}{\partial r}(b)$

First we consider all the $l \neq 1$ terms and show they vanish

For $l \neq 1$,

$$(2) \Rightarrow \quad A_\ell' b^\ell + \frac{B_\ell'}{b^{\ell+1}} = \frac{B_\ell}{b^{\ell+1}} \qquad (i)$$

$$(4) \Rightarrow \quad \epsilon \ell A_\ell' b^{\ell-1} - \frac{\epsilon(\ell+1) B_\ell'}{b^{\ell+2}} = -\frac{(\ell+1) B_\ell}{b^{\ell+2}}$$

multiply (2) by $\frac{\ell+1}{b}$ and add to get

$$\left[(\ell+1) b^{\ell-1} + \epsilon \ell b^{\ell-1}\right] A_\ell' + \left[\frac{\ell+1}{b^{\ell+2}} - \frac{\epsilon(\ell+1)}{b^{\ell+2}}\right] B_\ell' = 0$$

$$A_\ell' = \frac{\ell+1}{b^{\ell+2}} \frac{1-\epsilon}{b^{\ell-1}(\ell+1+\epsilon\ell)} B_\ell'$$

$$A_\ell' = \frac{\ell+1}{b^{2\ell+1}} \frac{1-\epsilon}{\ell+1+\epsilon\ell} B_\ell' \qquad (ii)$$

$$(1) \Rightarrow \quad A_\ell a^\ell = A_\ell' a^\ell + \frac{B_\ell'}{a^{\ell+1}}$$ substitute in

$$A_\ell = \left[\frac{\ell+1}{b^{2\ell+1}} \frac{1-\epsilon}{\ell+1+\epsilon\ell} + \frac{1}{a^{2\ell+1}}\right] B_\ell' \qquad (iii)$$

$$(3) \Rightarrow \quad \ell A_\ell a^{\ell-1} = \epsilon \ell A_\ell' a^{\ell-1} - \frac{\epsilon(\ell+1) B_\ell'}{a^{\ell+2}}$$

for $\ell = 0 \Rightarrow B_0' = 0$ then (iii) $\Rightarrow A_0 = 0$ and (ii) $\Rightarrow A_0' = C$
and (i) $\Rightarrow B_0 = 0$ so all $\ell = 0$ coefficients $= 0$

For $\begin{Bmatrix} l \neq 1 \\ l \neq 0 \end{Bmatrix}$, divide by $l a^{l-1}$ and substitute in (ii) to get

$$A_l = \epsilon A_l' - \frac{\epsilon(l+1)}{l a^{2l+1}} B_l'$$

$$A_l = \epsilon \left[ \frac{l+1}{b^{2l+1}} \frac{1-\epsilon}{l+1+\epsilon l} - \frac{(l+1)}{l a^{2l+1}} \right] B_l' \quad \text{(iv)}$$

Compare above with (iii).

(iii) is of the form $A_l = \gamma B_l'$ \
(iv) is of the form $A_l = \gamma' B_l'$ } with $\gamma \neq \gamma'$

only solution is $\underline{A_l = B_l' = A_l' = B_l = 0}$

<u>So only the $l=1$ coefficients do not vanish</u>

<u>For $l=1$</u>

(2) $A_1' b + \dfrac{B_1'}{b^2} = -E_0 b + \dfrac{B_1}{b^2}$

(4) $\epsilon A_1' - 2\epsilon \dfrac{B_1'}{b^3} = -E_0 - \dfrac{2B_1}{b^3}$

(1) $A_1 a = A_1' a + \dfrac{B_1'}{a^2} \Rightarrow A_1 = A_1' + \dfrac{B_1'}{a^3}$

(3) $A_1 = \epsilon A_1' - \dfrac{2\epsilon B_1'}{a^3}$

Substitute (3) into (1) to get

$$\epsilon A_1' - \frac{2\epsilon B_1'}{a^3} = A_1' + \frac{B_1'}{a^3}$$

$$\Rightarrow \quad A_1' = \frac{1}{a^3}\frac{(1+2\epsilon)}{\epsilon - 1} B_1' \qquad (*)$$

multiply (2) by $\frac{2}{b}$ ad add to (4)

$$\Rightarrow \quad (\epsilon+2)A_1' + \left(\frac{2}{b^3} - \frac{2\epsilon}{b^3}\right)B_1' = -3E_0$$

$$(\epsilon+2)A_1' + \frac{2}{b^3}(1-\epsilon)B_1' = -3E_0$$

Substitute from($*$) into above to get

$$\left[\frac{1}{a^3}\frac{\epsilon+2}{\epsilon-1}(1+2\epsilon) + \frac{2}{b^3}(1-\epsilon)\right]B_1' = -3E_0$$

$$B_1' = \frac{-3E_0}{\frac{1}{a^3}\frac{(\epsilon+2)(1+2\epsilon)}{\epsilon-1} + \frac{2}{b^3}(1-\epsilon)}$$

$$= \frac{-3E_0 a^3 b^3 (\epsilon-1)}{b^3(\epsilon+2)(1+2\epsilon) + 2a^3(1-\epsilon)(\epsilon-1)}$$

$$= \frac{-3E_0 a^3 b^3 (\epsilon-1)}{-2a^3(\epsilon-1)^2 + b^3(\epsilon+2)(1+2\epsilon)}$$

$$\boxed{B_1' = \frac{3E_0 a^3 b^3 (\epsilon-1)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}}$$

from (*)

$$A_1' = \frac{1}{a^3}\left(\frac{1+2\epsilon}{\epsilon-1}\right) B_1'$$

$$\boxed{A_1' = \frac{3E_0 b^3 (1+2\epsilon)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}}$$

$$(1) \Rightarrow \quad A_1 = A_1' + \frac{B_1'}{a^3} = \left[\frac{1}{a^3}\left(\frac{1+2\epsilon}{\epsilon-1}\right) + \frac{1}{a^3}\right] B_1'$$

$$= \left(\frac{1+2\epsilon}{\epsilon-1} + 1\right] \frac{3E_0 b^3(\epsilon-1)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}$$

$$\boxed{A_1 = \frac{9\epsilon E_0 b^3}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}}$$

$$(2) \Rightarrow \quad B_1 = A_1' b^3 + B_1' + E_0 b^3$$

$$= \left[\frac{b^3}{a^3}\,\frac{1+2\epsilon}{\epsilon-1} + 1\right] B_1' + E_0 b^3$$

$$= \frac{b^3(1+2\epsilon) + a^3(\epsilon-1)}{a^3(\epsilon-1)} \cdot \frac{3E_0 a^3 b^3(\epsilon-1)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}$$

$$+ E_0 b^3$$

$$= \frac{E_0 a^3 b^3(\epsilon-1)\left[3b^3(1+2\epsilon) + 3a^3(\epsilon-1) + 2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)\right]}{a^3(\epsilon-1)\left[2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)\right]}$$

$$B_1 = \frac{E_0 a^3 b^3 (\epsilon-1)}{a^3(\epsilon-1)} \; \frac{b^3(1+2\epsilon)(1-\epsilon) + a^3(\epsilon-1)(1+2\epsilon)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}$$

$$\boxed{B_1 = \frac{E_0 \, b^3 (1+2\epsilon)(\epsilon-1)(a^3-b^3)}{2a^3(\epsilon-1)^2 - b^3(\epsilon+2)(1+2\epsilon)}}$$

So we have solved for all the unknown coefficients

$\phi_I = A_1 r \cos\theta$ uniform $\vec{E}$ inside $\vec{E}_{in} = -A_1 \hat{z}$

$\phi_{II} = A_1' r \cos\theta + \frac{B_1'}{r^2} \cos\theta$

$\phi_{III} = -E_0 r \cos\theta + \frac{B_1}{r^2}\cos\theta$

uniform + dipole correction

electric field given by $\vec{E} = -\vec{\nabla}\phi = E_r \hat{r} + E_\theta \hat{\theta}$

with $E_r = -\frac{\partial \phi}{\partial r}$ and $E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

1

Check: if $\epsilon \to \infty$, we have a conductor.

As expected, the above formulas give $\underbrace{A_1 = A_1' = B_1' = 0}_{\text{no field inside}}$

and

$B_1 = E_0 b^3$ dipole correction to uniform $\vec{E}$ outside

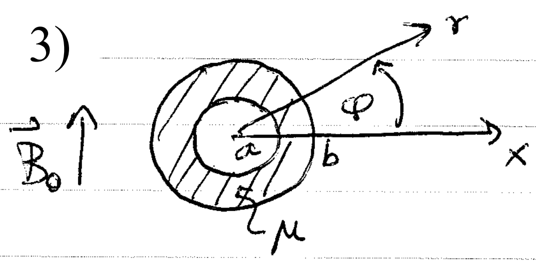
2 if $a \to 0$ (solid dielectric sphere) then

$B_1' = 0$, $A_1' = -\frac{3E_0}{\epsilon+2}$, $B_1 = E_0 b^3 \left(\frac{\epsilon-1}{\epsilon+2}\right)$

$\Rightarrow \vec{E}$ constant inside dielectric

agrees with solution in Jackson, Eq (4.54)

3)




cylinder $\parallel \hat{z}$

uniform applied field $\vec{B}_0 = B_0 \hat{y}$

Region I: $r<a$

II: $a<r<b$

III: $b<r$

Since free current $\vec{J}=0$ everywhere we can use magnetic scalar potential

$$\vec{\nabla}\times\vec{H}=0 \Rightarrow \vec{H} = -\vec{\nabla}\phi_M \quad , \quad \vec{B} = \mu\vec{H} \text{ in medium}$$

From the cylindrical symmetry we have (see separation of variables in polar coords)

$$\phi_M = A_0 + B_0 \ln r + \sum_{n=1}^{\infty}\left[A_n r^n \sin(n\varphi+\alpha_n) + \frac{B_n}{r^n}\sin(n\varphi+\beta_n)\right]$$

<u>Boundary conditions are</u>

normal component $\vec{B}$ continuous

$$\Rightarrow \quad ① \quad \frac{\partial \phi_M^{I}}{\partial r}(a) = \mu \frac{\partial \phi_M^{II}}{\partial r}(a)$$

$$② \quad \mu\frac{\partial \phi_M^{II}}{\partial r}(b) = \frac{\partial \phi_M^{III}}{\partial r}(b)$$

tangential component $\vec{H}$ continuous (no free surface current)

$$③ \quad \frac{\partial \phi_M^{I}}{\partial \varphi}(a) = \frac{\partial \phi_M^{II}}{\partial \varphi}(a)$$

$$④ \quad \frac{\partial \phi_M^{II}}{\partial \varphi}(b) = \frac{\partial \phi_M^{III}}{\partial \varphi}(b)$$

We expect the following forms

$$\phi_M^{I}(\vec{r}) = A_0 + \sum_{n=1}^{\infty} A_n r^n \sin(n\varphi + \alpha_n)$$

(no $\frac{B_n}{r^n}$ terms since $\phi_M$ should be finite at $r=0$)

$$\phi_M^{II}(\vec{r}) = A_0' + B_0' \ln r + \sum_{n=1}^{\infty} \left[ A_n' r^n \sin(n\varphi + \alpha_n') + \frac{B_n'}{r^n} \sin(n\varphi + \beta_n') \right]$$

$$\phi_M^{III}(\vec{r}) = -B_0 r \sin\varphi + \sum_{n=1}^{\infty} \frac{B_n}{r^n} \sin(n\varphi + \beta_n)$$

(no $A_n r^n$ terms or $B_0 \ln r$ term as the deviation from the applied field should vanish as $r \to \infty$)

To match angular dependencies one needs $\alpha_n = \alpha_n' = \beta_n' = \beta_n$

Then if one considered the $n \neq 1$ terms, one would find that eqns. ① – ④ give 4 homogeneous (no source term) linear equations for $A_n, A_n', B_n', B_n$. The equations are independent $\Rightarrow$ only solution will be $A_n = A_n' = B_n' = B_n$.

[Can express these 4 linear equations in matrix form:

$$(M)\begin{pmatrix} A_n \\ A_n' \\ B_n' \\ B_n \end{pmatrix} = 0.$$

eqns independent $\Rightarrow \det M = 0$

$\Rightarrow$ only solution is $\begin{pmatrix} A_n \\ A_n' \\ B_n' \\ B_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$]

So only the $n=1$ term is present.

To match angular dependencies with applied field, one needs $\alpha_1 = \alpha_1' = \beta_1' = \beta_1 = 0$

Apply boundary conditions to get

$$③ \Rightarrow A_1 a = A_1' a + \frac{B_1'}{a} \Rightarrow A_1 = A_1' + \frac{B_1'}{a^2}$$

$$① \Rightarrow A_1 = \mu A_1' - \mu \frac{B_1'}{a^2}$$

subtract ① from ③ $\Rightarrow$ $A_1'(1-\mu) + \frac{B_1'}{a^2}(1+\mu) = 0$

$$A_1' = \frac{B_1'}{a^2}\frac{(1+\mu)}{(\mu-1)} \qquad (*)$$

$$④ \Rightarrow A_1' b + \frac{B_1'}{b} = -B_0 b + \frac{B_1}{b} \Rightarrow A_1' + \frac{B_1'}{b^2} = -B_0 + \frac{B_1}{b^2}$$

$$② \Rightarrow \mu A_1' - \mu \frac{B_1'}{b^2} = -B_0 - \frac{B_1}{b^2}$$

add ④ + ② $\Rightarrow$ $A_1'(1+\mu) + \frac{B_1'}{b^2}(1-\mu) = -2B_0$

substitute $(*)$ in above

$$B_1' \left[ \frac{(1+\mu)^2}{a^2(\mu-1)} + \frac{1-\mu}{b^2} \right] = -2B_0$$

$$B_1' = \frac{-2B_0}{\frac{(1+\mu)^2}{a^2(\mu-1)} + \frac{1-\mu}{b^2}}$$

$$\boxed{B_1' = \frac{-2B_0 a^2 b^2 (\mu-1)}{b^2(\mu+1)^2 - a^2(\mu-1)^2}}$$

From (*) $A_1' = \frac{B_1'}{a^2}\frac{1+\mu}{\mu-1}$.

$$\boxed{A_1' = \frac{-2B_0 b^2(\mu+1)}{b^2(\mu+1)^2 - a^2(\mu-1)^2}}$$

From (3) $A_1 = A_1' + \frac{B_1'}{a^2} = \frac{-2B_0 b^2(\mu+1) - 2B_0 b^2(\mu-1)}{b^2(\mu+1)^2 - a^2(\mu-1)^2}$

$$\boxed{A_1 = \frac{-4B_0 b^2 \mu}{b^2(\mu+1)^2 - a^2(\mu-1)^2}}$$

From (4) $B_1 = A_1' b^2 + B_1' + B_0 b^2$

$$= \frac{-2B_0 b^4(\mu+1) - 2B_0 a^2 b^2(\mu-1)}{b^2(\mu+1)^2 - a^2(\mu-1)^2} + B_0 b^2$$

$$= \frac{-2B_0 b^4(\mu+1) - 2B_0 a^2 b^2(\mu-1) + B_0 b^4(\mu+1)^2 - B_0 a^2 b^2(\mu-1)^2}{b^2(\mu+1)^2 - a^2(\mu-1)^2}$$

$$= \frac{B_0 b^4(\mu+1)[\mu+1-2] - B_0 a^2 b^2(\mu-1)[\mu-1+2]}{b^2(\mu+1)^2 - a^2(\mu-1)^2}$$

$$B_1 = \frac{B_0 b^4(\mu+1)(\mu-1) - B_0 a^2 b^2(\mu-1)(\mu+1)}{b^2(\mu+1)^2 - a^2(\mu-1)^2}$$

$$\boxed{B_1 = \frac{B_0 b^2(\mu^2-1)(b^2-a^2)}{b^2(\mu+1)^2 - a^2(\mu-1)^2}}$$

we have solved for all the unknown coefficients!

$\phi^{I}(\vec{r}) = A_1 r \sin\varphi$ uniform field $\vec{B} = -A_1 \hat{y}$ inside shell

$\phi_M^{II}(\vec{r}) = A_1' r \sin\varphi + \frac{B_1'}{r} \sin\varphi$

$\phi_M^{III}(\vec{r}) = -B_0 r \sin\varphi + \frac{B_1}{r} \sin\varphi$

$$\vec{B} = -\mu \vec{\nabla}\phi_M = -\mu\left[ B_r \hat{r} + B_\varphi \hat{\varphi} \right]$$

$$B_r = -\mu \frac{\partial \phi_M}{\partial r}$$

$$B_\varphi = -\frac{\mu}{r}\frac{\partial \phi_M}{\partial \varphi}$$

Checks: ① $\mu \to \infty \Rightarrow A_1' = B_1' = A_1 = 0$

$B_1 = B_0 b^2$ (indep of $a$)

magnetic field completely shielded from inside of shell.

② $a \to 0$ solid permeable cylinder

$A_1' = \frac{-2B_0}{\mu+1}$, $B_1' = 0$

$\Rightarrow \vec{H}$ is constant inside cylinder

$B_1 = B_0 b^2 \left(\frac{\mu-1}{\mu+1}\right)$