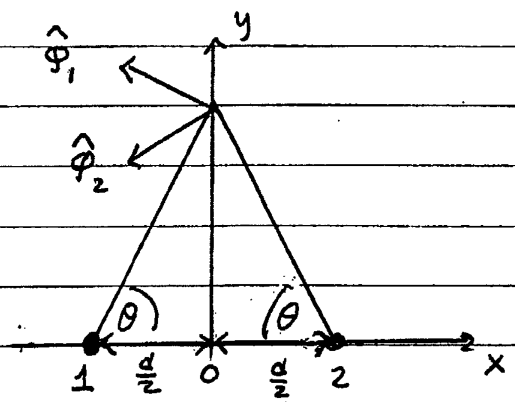
PHY 415 Solutions Problem Set 6

1) Top view of the wires



wires out of plane of page

At the point $(x=0, y, z)$ on the plane that bisects the space between the two wires,

$$\vec{B} = \frac{2I_1}{c\sqrt{(\frac{d}{2})^2+y^2}}\hat{\varphi}_1 + \frac{2I_2}{c\sqrt{(\frac{d}{2})^2+y^2}}\hat{\varphi}_2$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are the polar angle unit vectors with respect to wires 1 and 2 (see diagram)

$$\hat{\varphi}_1 = \cos\theta\,\hat{y} - \sin\theta\,\hat{x}$$
$$\hat{\varphi}_2 = -\cos\theta\,\hat{y} - \sin\theta\,\hat{x}$$

$$\cos\theta = \frac{d/2}{\sqrt{(d/2)^2+y^2}} \qquad \sin\theta = \frac{y}{\sqrt{(\frac{d}{2})^2+y^2}}$$

So on the yz plane separating the two wires

$$\vec{B}(y) = \frac{2\cos\theta}{c\sqrt{(\frac{d}{2})^2+y^2}}(I_1-I_2)\,\hat{y} - \frac{2\sin\theta}{c\sqrt{(\frac{d}{2})^2+y^2}}(I_1+I_2)\,\hat{x}$$

$$\vec{B} = \frac{d\,(I_1-I_2)}{c\left[(\frac{d}{2})^2+y^2\right]}\hat{y} - \frac{2y\,(I_1+I_2)}{c\left[(\frac{d}{2})^2+y^2\right]}\hat{x}$$

when $I_1 = I_2$, 1st term vanishes! $\vec{B} = B\hat{x}$

when $I_1 = -I_2$, 2nd term vanishes! $\vec{B} = B\hat{y}$

Maxwell stress tensor

when $\vec{E}=0$ $\quad T_{ij} = \frac{1}{4\pi}\left\{ B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right\}$

we want $-\int da\, \overleftrightarrow{T}\cdot\hat{n} = -\int dz \int dy\, \overleftrightarrow{T}\cdot\hat{x}$

$\overleftrightarrow{T}\cdot\hat{x} = T_{xx}\,\hat{x} + T_{yx}\,\hat{y} + T_{zx}\,\hat{z}$

Since $\vec{B} = \dfrac{d(I_1 - I_2)}{c\left[y^2 + (\frac{d}{2})^2\right]}\,\hat{y} - \dfrac{2y(I_1+I_2)}{c\left[y^2+(\frac{d}{2})^2\right]}\,\hat{x}$

has only x and y components, then $T_{xz} = T_{yz} = 0$

$$T_{ij} = \frac{1}{4\pi}\begin{pmatrix} \frac{1}{2}B_x^2 - \frac{1}{2}B_y^2 & B_x B_y & 0 \\ B_y B_x & \frac{1}{2}B_y^2 - \frac{1}{2}B_x^2 & 0 \\ 0 & 0 & -\frac{1}{2}(B_x^2+B_y^2) \end{pmatrix}$$

Consider the term $\int dy\, T_{yx}$

$$\int_{-\infty}^{\infty} dy\, T_{yx} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dy\, B_x B_y = \frac{-1}{4\pi}\int_{-\infty}^{\infty} dy\, \frac{d(I_1-I_2)\,2y(I_1+I_2)}{c^2\left[y^2+(\frac{d}{2})^2\right]^2}$$

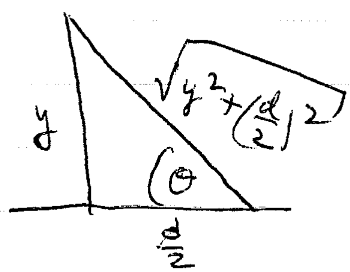
$= 0$ since integrand is odd in y.

So the only term that contributes to $\int da\, \overleftrightarrow{T}\cdot\hat{n}$ is $\hat{x}\int da\, T_{xx}$

$$\int da\, T_{xx} = \frac{1}{8\pi}\int dz \int dy \left[B_x^2 - B_y^2 \right]$$

$$= \frac{1}{8\pi c^2}\int dz \int_{-\infty}^{\infty} dy \left[\frac{4y^2 (I_1+I_2)^2}{(y^2+(\frac{d}{2})^2)^2} - \frac{d^2(I_1-I_2)^2}{(y^2+(\frac{d}{2})^2)^2} \right]$$

To do the integrals do a trig substitution of variables



$$\Rightarrow \frac{y^2}{\left[y^2+(\frac{d}{2})^2\right]} = \sin^2\theta$$

$$\frac{1}{\left[y^2+(\frac{d}{2})^2\right]} = \left(\frac{2}{d}\right)^2 \cos^2\theta$$

$$y = \frac{d}{2}\tan\theta \Rightarrow dy = \frac{d}{2}\frac{d\tan\theta}{d\theta}d\theta = \frac{d}{2}\frac{d\theta}{\cos^2\theta}$$

$$\int da\, T_{xx} = \frac{1}{8\pi c^2}\int dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{d}{2\cos^2\theta}\left[4(I_1+I_2)^2 \sin^2\theta \left(\frac{2}{d}\right)^2\cos^2\theta - d^2(I_1-I_2)^2\left(\frac{2}{d}\right)^4\cos^4\theta \right]$$

$$= \frac{1}{8\pi c^2}\frac{8}{d}\int dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[(I_1+I_2)^2 \sin^2\theta - (I_1-I_2)^2\cos^2\theta \right]$$

$$= \frac{1}{\pi d c^2}\int dz\ \frac{\pi}{2}\left[(I_1+I_2)^2 - (I_1-I_2)^2 \right]$$

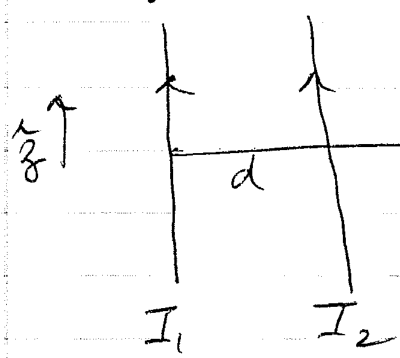
$$= \frac{L_z}{2dc^2}\left[I_1^2 + I_2^2 + 2I_1I_2 - (I_1^2+I_2^2-2I_1I_2) \right]$$

$$= \frac{2L_z}{d\,c^2} I_1 I_2$$

so $$\boxed{-\int da\, \overleftrightarrow{T}\cdot\hat{n} = -2 L_z \frac{I_1 I_2}{dc^2}\,\hat{x}}$$

for all values of I_1 and I_2

The above is exactly equal to the magnetostatic force on wire 2 due to the magnetic field of wire 1.



$$\vec{F}_2 = L_z \frac{\vec{I}_2}{c} \times \vec{B}_1$$

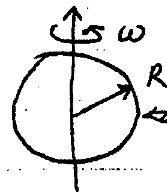
$$= L_z \frac{I_2}{c}\hat{z} \times \left(\frac{2I_1}{dc}\hat{y}\right)$$

$$= -2\frac{L_z I_1 I_2}{dc^2}\hat{x}$$

If I_1 and I_2 are in the same direction, the force is attractive.
If I_1 and I_2 are in opposite directions, the force is repulsive

$-\int da\, \overleftrightarrow{T}\cdot\hat{n}$ is the flux of electromagnetic momentum passing through the plane which is the total electromagnetic force on the plane (force = momentum / time) pushing from side 1 to side 2. We would get the same result is we computed $-\int da\, \overleftrightarrow{T}\cdot\hat{n}$ over any y-z plane inbetween the two wires (not just midway). We can view this like an electromagnetic pressure that fills space and transmits the force from wire 1 to wire 2

2)

$\hat{z}\uparrow$ 

← net charge e uniformly distributed on surface

first we find $\vec{E}$ + $\vec{B}$ fields. The above is an electrostatic and magnetostatic situation, since $\frac{\partial \rho}{\partial t} = 0$ and $\vec{\nabla}\cdot\vec{J} = 0$

$$\boxed{\vec{E}(\vec{r}) = \begin{cases} \frac{e}{r^2}\hat{r} & r > R \\ 0 & r < R \end{cases}}$$

outside spherical shell, $\vec{E}$ looks just like pt charge at the origin

From problem Set 5, problem 1(b), we had

$r \leq R$ $\quad \vec{B} = \frac{8\pi\sigma}{3c}\omega R\,\hat{z}$ $\qquad$ here $\sigma = \frac{e}{4\pi R^2}$

$$\boxed{\vec{B} = \frac{2e\omega}{3cR}\hat{z}}$$

$r \geq R$ $\quad \vec{B} = \frac{4}{3c}\pi\sigma\omega R^4 \, \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}$

$$\boxed{\vec{B} = \frac{e\omega R^2}{3c\,r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

This is just a dipole field with total dipole moment $|\vec{m}| = \frac{e\omega R^2}{3c} = \frac{4\pi\omega\sigma R^4}{3c}$

a) The total electromagnetic energy is

$$W = \frac{1}{8\pi}\int d^3r\left[E^2 + B^2\right]$$

Since this is an electrostatic and magnetostatic situation it is easier to use

$$W = \frac{1}{2}\int d^3r\, \rho\phi + \frac{1}{2c}\int d^3r\, \vec{j}\cdot\vec{A}$$

since then the integrals are confined to the shell $r = R$ where ρ and $\vec{j}$ are not zero.

We have for electrostatic potential

$$\phi(r) = \begin{cases} \dfrac{e}{r} & r \geq R \\[2mm] \dfrac{e}{R} & r \leq R \end{cases}$$

$$W_{elec} = \frac{1}{2}\int d^3r\, \rho\,\phi = \frac{1}{2}\, 4\pi R^2 \sigma\, \phi(R) = \frac{1}{2}\frac{e^2}{R}$$

For the magnetostatic vector potential we have

$$\vec{A}(\vec{r}) = \frac{\vec{m}\times\hat{r}}{r^2} = \frac{m\sin\theta}{r^2}\,\hat{\varphi} \qquad r \geq R$$

since $\vec{B}$ is a pure dipole field. The magnetic dipole moment is

$$\vec{m} = \frac{4\pi}{3c}\,\omega\sigma R^4$$

$$\vec{A}(r=R) = \frac{4\pi}{3c}\,\omega\sigma R^2 \sin\theta\,\hat{\varphi}$$

$$W_{mag} = \frac{1}{2c}\int d^3r\,\vec{J}\cdot\vec{A} = \frac{1}{2}\int da\;\vec{K}\cdot\vec{A}$$

$$\vec{K} = \sigma\omega R\sin\theta\,\hat{\varphi} \quad \text{(see prob 1a, Set 5)}$$

$$W_{mag} = \frac{1}{2c}\int da\;\frac{4\pi}{3c}\,\omega^2\sigma^2 R^3 \sin^2\theta$$

$$= \frac{2\pi R^2}{2c}\,\frac{4\pi}{3c}\,\omega^2\sigma^2 R^3 \int_0^\pi d\theta\,\sin\theta\,\sin^2\theta$$

$$= \frac{4\pi^2}{3c^2}\,\omega^2\sigma^2 R^5 \underbrace{\int_0^\pi d\theta\,\sin\theta\left[1-\cos^2\theta\right]}$$

$$= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi$$

$$= \left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right) = \frac{4}{3}$$

$$W_{mag} = \frac{(4\pi)^2}{9c^2}\,\omega^2\sigma^2 R^5 = \frac{(4\pi)^2}{9c^2}\,\omega^2\,\frac{e^2 R^5}{(4\pi R^2)^2}$$

$$= \frac{\omega^2 e^2 R}{9c^2}$$

$$W = W_{elec} + W_{mag}$$

$$\boxed{W = \frac{1}{2}\frac{e^2}{R}\left[1 + \frac{2}{9}\,\frac{R^2\omega^2}{c^2}\right]}$$

b) Angular momentum density

$$\vec{\mathcal{L}} = \vec{r} \times \vec{\Pi} = \frac{1}{4\pi c}\, \vec{r} \times (\vec{E} \times \vec{B})$$

$\mathcal{L} = 0$ inside shell since $\vec{E} = 0$ inside

outside shell

$$\vec{E} \times \vec{B} = \frac{e}{r^2}\, \hat{r} \times \left[\frac{e\omega R^2}{3c\, r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right]$$

$$= \frac{e^2 \omega R^2}{3c\, r^5} \sin\theta\, \hat{\varphi} \qquad \text{as } \hat{r} \times \hat{\theta} = \hat{\varphi}$$

$$\vec{\mathcal{L}} = \frac{1}{4\pi c}\, \vec{r} \times (\vec{E} \times \vec{B}) = -\frac{e^2 \omega R^2}{12\pi c^2 r^4} \sin\theta\, \hat{\theta} \qquad \text{as } \hat{r} \times \hat{\varphi} = -\hat{\theta}$$

$$\boxed{\vec{\mathcal{L}} = -\frac{e^2 \omega R^2}{12\pi c^2 r^4} \sin\theta\, \hat{\theta}} \quad r \geq R, \quad \mathcal{L} = 0, \quad r \leq R$$

integrate over all space to get total angular momentum

$$\vec{L} = \int d^3r\, \vec{\mathcal{L}} = 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2\, \vec{\mathcal{L}}$$

By symmetry we only need to integrate $\mathcal{L}_z$ since other components will vanish.

$$\hat{\theta}\cdot\hat{z} = -\sin\theta \Rightarrow \mathcal{L}_z = \frac{e^2\omega R^2}{12\pi c^2 r^4}\sin^2\theta$$

$$L_z = \frac{e^2\omega R^2}{12\pi c^2}\, 2\pi \underbrace{\int_0^\pi d\theta \sin^3\theta}_{\frac{4}{3}} \underbrace{\int_R^\infty dr\, \frac{1}{r^2}}_{\frac{1}{R}}$$

since $\mathcal{L} = 0$ for $r \leq R$

$$\boxed{L_z = \frac{2}{9}\frac{e^2\omega R}{c^2}}$$

c)

$$L_z = \frac{2}{9}\frac{e^2\omega R}{c^2} = \frac{\hbar}{2}$$

$$\Rightarrow \frac{\omega R}{c} = \frac{9}{4}\frac{\hbar c}{e^2}$$

use $\frac{\hbar c}{e^2} = 137 = \frac{1}{\alpha}$ fine structure constant

$$\frac{\omega R}{c} = \left(\frac{9}{4}\right)(137) = 308.25$$

so the velocity at the equator is

$$\boxed{\omega R = 308.25\, c}$$

much faster than the speed of light!

Clearly this model cannot be correct!

d) $$W = \frac{1}{2}\frac{e^2}{R}\left[1+\frac{2}{9}\left(\frac{R\omega}{c}\right)^2\right]$$

$$= \frac{1}{2}\frac{e^2}{R}\left[1+\frac{2}{9}\left(\frac{9}{4}\right)^2(137)^2\right]$$

$$W = \frac{1}{2}\frac{e^2}{R}\left[2.1116\times10^4\right] = mc^2$$

$$R = \frac{1}{2}\frac{e^2}{mc^2}\left(2.1116\times10^4\right)$$

if a_0 is Bohr radius

$$\frac{R}{a_0} = \frac{1}{2}\frac{e^2}{a_0}\frac{1}{mc^2}\left(2.1116\times10^4\right)$$

use $\frac{e^2}{2a_0} = 13.6$ eV , $mc^2 = 0.511\times10^6$ eV

so $$\frac{R}{a_0} = \frac{(13.6)(2.1116\times10^4)}{0.511\times10^6} = \boxed{0.562 = \frac{R}{a_0}}$$

electron radius is $\sim\frac{1}{2}$ Bohr radius! Clearly not correct! Much too big!

Putting in numbers : $R = (0.562)(0.529\times10^{-8}\text{ cm}) = 0.297$ Å

$$\omega = (308.25)\frac{(3\times10^{10}\text{ cm/sec})}{(0.297\times10^{-8}\text{ cm})} = 3.11\times10^{21}\ \frac{\text{rads}}{\text{sec}}$$

Problem Set 8, problem 2

In my solution to problem 2 on Problem Set 8, I computed the electromagnetic energy of the spinning charged sphere,

$$W = \frac{1}{8\pi}\int d^3r\ \left[|\mathbf{E}|^2 + |\mathbf{B}|^2\right] \tag{1}$$

as

$$W = \frac{1}{2}\int d^3r\,\rho\phi + \frac{1}{2c}\int d^2r\,\mathbf{j}\cdot\mathbf{A} \tag{2}$$

which should be correct in an electrostatic and magnetostatic situation such as we have in the problem.

I then wrote the above as,

$$W = \frac{1}{2}\oint_S da\,\sigma\phi + \frac{1}{2c}\oint_S da\,\mathbf{K}\cdot\mathbf{A} \qquad \text{with } S \text{ the surface of the sphere of radius } R \tag{3}$$

since all the charge and current are restricted to the surface of the sphere. Here $\sigma = e/(4\pi R^2)$ is the surface charge density, and $\mathbf{K} = \sigma\omega R\sin\theta\,\hat{\boldsymbol{\varphi}}$ is the surface current density.

But you may have wondered, is that OK to do? When I compute the above surface integrals, should I use the values of ϕ and $\mathbf{A}$ just on the outside of the surface of the sphere, or should I use their values just on the inside of the surface of the sphere? There could possibly be a difference if either ϕ or $\mathbf{A}$ were discontinuous at the surface.

The answer is, YES! it is OK to do. One reason is that one can choose potentials ϕ and $\mathbf{A}$ that are continuous at the surface S, so it does not matter if we use the values just outside or just inside. We can also directly compute the energy W using Eq. (1) and confirm we get the same answer.

Electrostatic Energy

For the electrostatic part of the problem, we know that ϕ can be chosen to be continuous at the surface. The electric field due to the uniform surface charge on the sphere is,

$$\mathbf{E} = \begin{cases} \dfrac{q}{r^2}\hat{\mathbf{r}} & r > R \quad \text{outside} \\[2ex] 0 & r < R \quad \text{inside} \end{cases} \tag{4}$$

and we can take as the potential

$$\phi = \begin{cases} \dfrac{q}{r} & r > R \quad \text{outside} \\[2ex] \dfrac{q}{R} & r < R \quad \text{inside} \end{cases} \tag{5}$$

You can confirm that the above satisfies $\mathbf{E} = -\boldsymbol{\nabla}\phi$. Since $\phi(r)$ is continuous at $r = R$, there is no problem defining the integral $\oint da\,\sigma\phi$. But we can also explicitly compute W_{elec} from $\mathbf{E}$,

$$W_{\text{elec}} = \frac{1}{8\pi}\int d^2r\,|\mathbf{E}|^2 = \frac{1}{8\pi}4\pi\int_R^\infty dr\,r^2E^2 = \frac{q^2}{2}\int_0^\infty dr\,\frac{1}{r^2} = \frac{q^2}{2R} \tag{6}$$

which is exactly the same as we found from $\frac{1}{2}\oint da\,\sigma\phi$.

Magnetostatic Energy

For the magnetostatic part of the problem, we have the magnetic field from problem 1 of Problem Set 5,

$$\mathbf{B} = \begin{cases} \dfrac{m}{r^3}[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}] & r > R \quad \text{outside} \\[2ex] \dfrac{2m}{R^3}\,\hat{\mathbf{z}} & r < R \quad \text{inside} \end{cases} \tag{7}$$

where the magnetic dipole moment is $\mathbf{m} = m\hat{\mathbf{z}}$ with $m = \dfrac{e\omega R^2}{3c}$. Outside the magnetic field is just like a point magnetic dipole with dipole moment $\mathbf{m}$, while inside the magnetic field is constant.

The vector potential for the field outside is therefore just the dipole vector potential,

$$\mathbf{A}^{\mathrm{out}} = \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^3} = \frac{m\sin\theta}{r^2}\,\hat{\boldsymbol{\varphi}} \tag{8}$$

To get the vector potential inside, we note that a constant magnetic field $B\hat{\mathbf{z}}$ can be given by the potential

$$\mathbf{A} = \frac{B}{2}[x\hat{\mathbf{y}} - y\hat{\mathbf{x}}] \qquad \Rightarrow \qquad \boldsymbol{\nabla}\times\mathbf{A} = B\hat{\mathbf{z}} \tag{9}$$

Now $\hat{\mathbf{z}}\times\mathbf{r} = \hat{\mathbf{z}}\times(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = (x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$, so we can rewrite the above as, $\mathbf{A} = \dfrac{B}{2}\,\hat{\mathbf{z}}\times\mathbf{r} = \dfrac{Br\sin\theta}{2}\,\hat{\boldsymbol{\varphi}}$. So we therefore have,

$$\mathbf{A}^{\mathrm{in}} = \frac{2m}{R^3}\,\frac{r\sin\theta}{2}\,\hat{\boldsymbol{\varphi}} = \frac{mr\sin\theta}{R^3}\,\hat{\boldsymbol{\varphi}} \tag{10}$$

Evaluating on the surface at $r = R$ we have,

$$\mathbf{A}^{\mathrm{out}} = \frac{m\sin\theta}{R^2}\,\hat{\boldsymbol{\varphi}}, \qquad \mathbf{A}^{\mathrm{in}} = \frac{mR\sin\theta}{R^3}\,\hat{\boldsymbol{\varphi}} = \frac{m\sin\theta}{R^2}\,\hat{\boldsymbol{\varphi}} \tag{11}$$

and so on the surface of the sphere $\mathbf{A}^{\mathrm{out}} = \mathbf{A}^{\mathrm{in}}$ and the vector potential is continuous. Hence there is no problem defining the integral $\oint da\,\mathbf{K}\cdot\mathbf{A}$. You might ask, what if we chose a different form for $\mathbf{A}^{\mathrm{in}}$ than that of Eq. (9). But we have shown in an earlier discussion question that $\int d^3r\,\mathbf{j}\cdot\mathbf{A}$ is independent of the gauge of $\mathbf{A}$, and one can show that the same is true of a surface term $\oint da\mathbf{K}\cdot\mathbf{A}$.

We can also check that we have the correct answer by directly computing W_{mag} from $\mathbf{B}$.

$$W_{\mathrm{mag}} = \frac{1}{8\pi}\int d^3r\,|\mathbf{B}|^2 = \frac{1}{8\pi}\int_{\mathrm{outside}} d^3r\,|\mathbf{B}^{\mathrm{out}}|^2 + \frac{1}{8\pi}\int_{\mathrm{inside}} d^3r\,|\mathbf{B}^{\mathrm{in}}|^2 \tag{12}$$

Since $\mathbf{B}^{\mathrm{in}} = (2m/R^3)\hat{\mathbf{z}}$ is constant, the contribution from the inside is,

$$W_{\mathrm{mag,in}} = \frac{1}{8\pi}\,\frac{4\pi R^3}{3}\left(\frac{2m}{R^3}\right)^2 = \frac{2m^2}{3R^3} \tag{13}$$

Since $\mathbf{B}^{\mathrm{out}} = m(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/r^3$, the contribution from the outside is,

$$W_{\mathrm{mag,out}} = \frac{1}{8\pi}m^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\int_R^{\infty}dr\,r^2\,\frac{4\cos^2\theta + \sin^2\theta}{r^6} \tag{14}$$

$$= \frac{1}{8\pi}m^2 2\pi\int_0^{\pi}d\theta\,\left(4\sin\theta\cos^2\theta + \sin^3\theta\right)\int_R^{\infty}dr\,\frac{1}{r^4} \tag{15}$$

$$= \frac{m^2}{4}\,\frac{1}{3R^3}\int_0^{\pi}d\theta\left(4\sin\theta\cos^2\theta + \sin\theta[1-\cos^2\theta]\right) \tag{16}$$

$$= \frac{m^2}{12R^3}\left[-\frac{4}{3}\cos^3\theta - \cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{m^2}{12R^3}\,[4] = \frac{m^2}{3R^3} \tag{17}$$

Note, $W_{\mathrm{mag,in}} = 2W_{\mathrm{mag,out}}$. So the total magnetostatic energy stored in the magnetic fields is,

$$W_{\mathrm{mag}} = W_{\mathrm{mag,in}} + W_{\mathrm{mag,out}} = \frac{2m^2}{3R^3} + \frac{m^2}{3R^3} = \frac{m^2}{R^3} \tag{18}$$

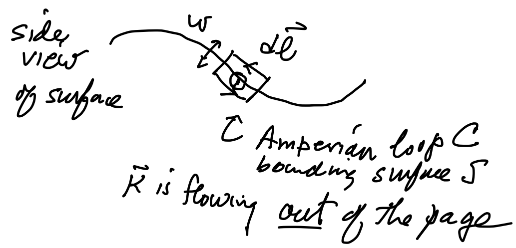
Using $m = e\omega R^2/3c$, we then get,

$$W_{\rm mag} = \frac{e^2\omega^2 R^4}{9c^2 R^3} = \frac{e^2\omega^2 R}{9c^2} \tag{19}$$

This is the same result as we obtained from $(1/2c)\oint da\,\mathbf{K}\cdot\mathbf{A}$.

General Comments

One can ask, is it just some peculiarity of this particular example that caused $\mathbf{A}^{\rm in} = \mathbf{A}^{\rm out}$ on the surface of the sphere? Might it be different for some other situation?



Let us consider the behavior of $\mathbf{A}$ at a surface current. Take a small Amperian loop C of length $d\ell$ parallel to the surface, and width w perpendicular to the surface. By Stokes law we can write for the magnetic flux through that loop,

$$\int_S da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_S da\,\hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \oint_C d\boldsymbol{d\ell}\cdot\mathbf{A} \tag{20}$$

where S is the surface bounded by the Amperian loop C and $\hat{\mathbf{n}}$ is the normal to that surface.

As we take $w\to 0$, the flux through the loop vanishes as $\mathbf{B}$ stays finite. The contribution of the sides of width w to the circulation of $\mathbf{A}$ around the loop also vanishes. We thus get,

$$0 = \boldsymbol{d\ell}\cdot(\mathbf{A}^{\rm above} - \mathbf{A}^{\rm below}) \tag{21}$$

Since $\boldsymbol{d\ell}$ can point in any direction within the tangent plane of the surface, we conclude that the tangential component of $\mathbf{A}$ must always be continuous at a surface current.

Now consider,

$$\int da\,\mathbf{K}\cdot\mathbf{A} \tag{22}$$

where the integral is over a surface containing the surface current $\mathbf{K}$. Since $\mathbf{K}$ necessarily lies in a direction tangent to the surface, it is only the tangential component of $\mathbf{A}$ that contributes to this integral. Since the tangential components of $\mathbf{A}^{\rm above}$ and $\mathbf{A}^{\rm below}$ must be equal, it therefore does not matter whether we use $\mathbf{A}^{\rm above}$ or $\mathbf{A}^{\rm below}$ when computing this integral – we must get the same answer in either case.