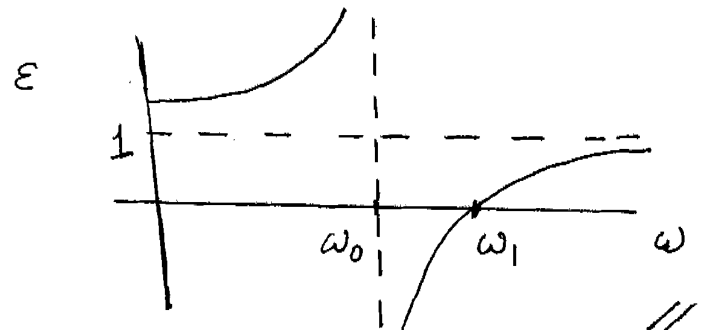
1) a) $\varepsilon(\omega) = 1 + \dfrac{\omega_p^2}{\omega_0^2-\omega^2} = \dfrac{\omega_0^2+\omega_p^2-\omega^2}{\omega_0^2-\omega^2}$

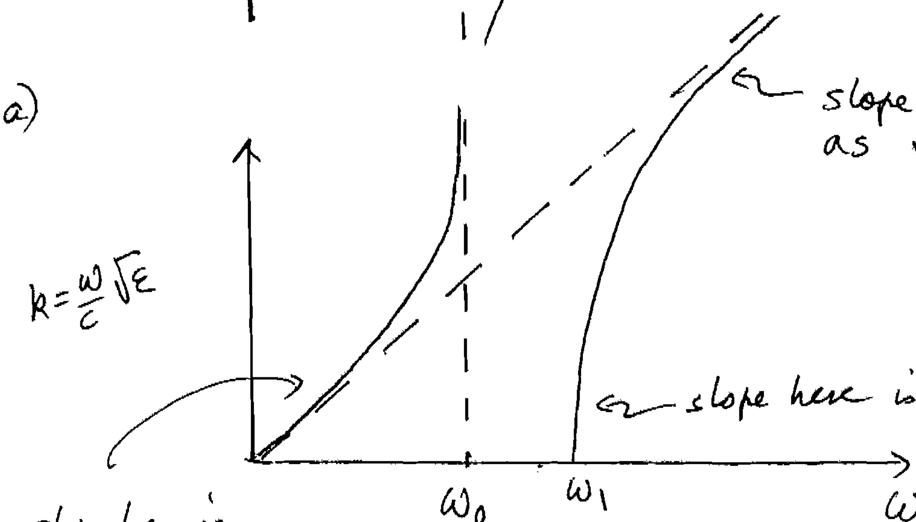
$k^2 = \dfrac{\omega^2}{c^2}\varepsilon(\omega)$ (take $\mu = 1$)

$k = \dfrac{\omega}{c}\sqrt{\varepsilon(\omega)}$



$\omega_1 = \sqrt{\omega_0^2+\omega_p^2}$ gives $\varepsilon = 0$

a)



slope here is $\frac{1}{c}$ as $\sqrt{\varepsilon} \to 1$ as $\omega \to \infty$

slope here is ∞ as $\varepsilon \to 0$ but $\dfrac{d\varepsilon}{d\omega}$ is finite

$\Rightarrow \dfrac{dk}{d\varepsilon} \sim \dfrac{\omega}{c}\dfrac{1}{\sqrt{\varepsilon}}\dfrac{d\varepsilon}{d\omega}$

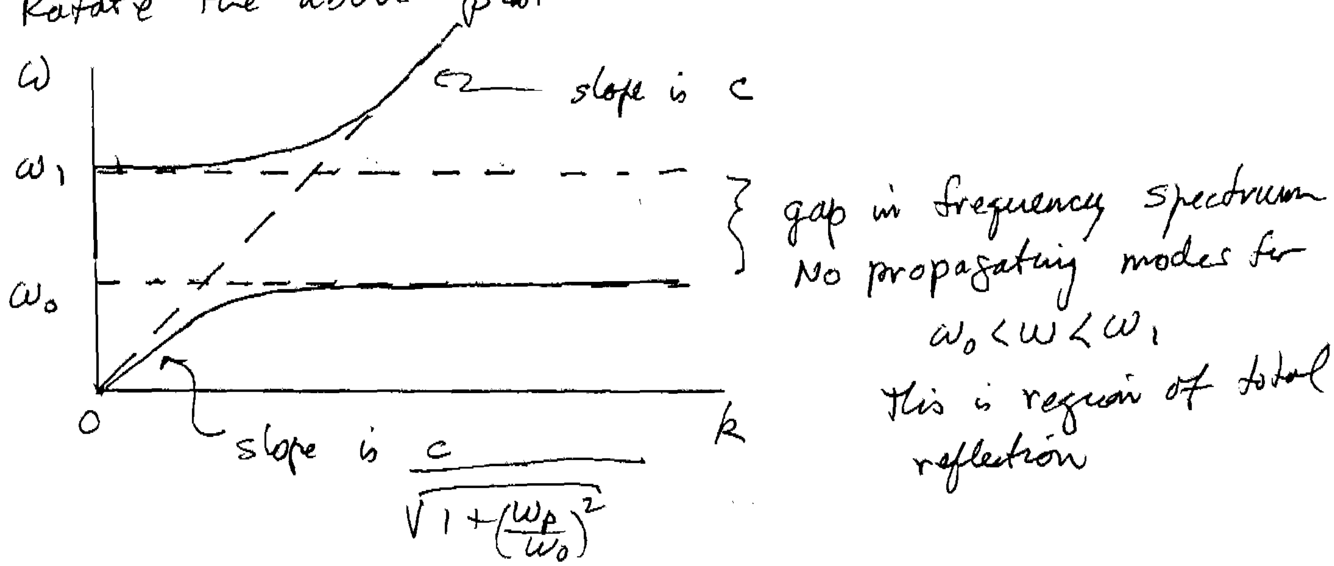
slope here is $> \frac{1}{c}$ as $\sqrt{\varepsilon} > 1$ as $\omega \to 0$

We see that at small ω, $k \approx \frac{\omega}{c}\sqrt{\varepsilon(0)}$ is linear in ω with phase velocity $v_p = \dfrac{c}{\sqrt{\varepsilon(0)}}$

$$v_p = \frac{c}{\sqrt{1+(\omega_p/\omega_0)^2}}$$

as $\omega \to \infty$, $\sqrt{\varepsilon(\omega)} \to 1$ and $k = \frac{\omega}{c}$, just as in a vacuum.

b) Rotate the above plot

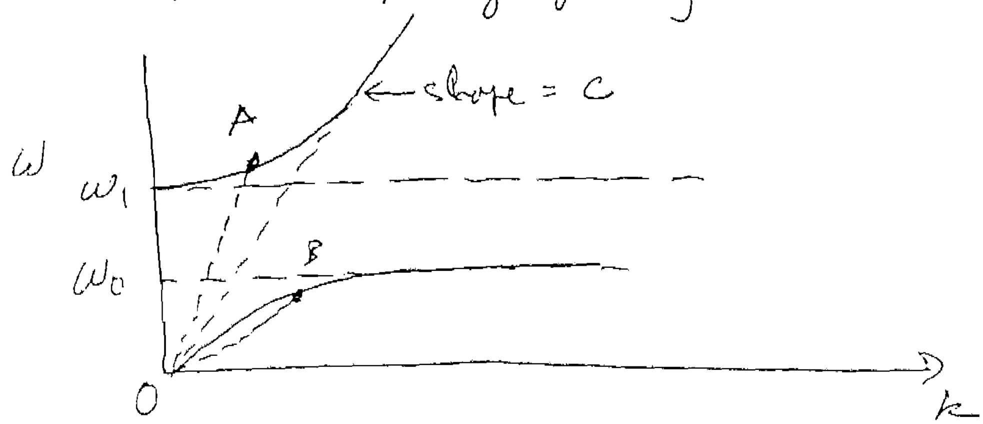


At small k, the lower branch is like a wave in a vacuum, ie linear dispersion relation $\omega \simeq v_p k$ with $v_p = c/\sqrt{1+(\omega_p/\omega_0)^2}$, while upper branch is roughly constant, $\omega \simeq \omega_1 + O(k^2)$

At large k, it is the upper branch that has linear dispersion relation, $\omega \approx ck$, while the lower branch is constant, $\omega \approx \omega_0$

At intermediate k the two modes cross over
These are called "polaritons" and represent a mixture of propagating wave and atomic oscillation

c) To see that $v_p > c$ for the upper mode, while $v_p < c$ for the lower mode, It is easiest to show this graphically



The phase velocity is $v_p = \frac{\omega}{c}$

For a point on the upper curve, for example A, the phase velocity is just the slope of the line $\overline{OA}$.

It is clear that the slope of this line is always greater than C as the point A varies on the curve.

Similarly, for a pt B on the lower curve, the phase velocity is the slope of the line $\overline{OB}$.

It is clear that this is always ~~below~~ less than the value $\dfrac{c}{\sqrt{1+\left(\frac{\omega_p}{\omega_0}\right)^2}} < c$ which is the slope as $\omega \to 0$.

So $v_p > c$ on upper curve

$v_p < c$ on lower curve

The group velocity $v_g = \frac{d\omega}{dk}$

Again it is easiest to see graphically that v_g, which is the tangent to the curve, is always less than c, for both upper & lower branches.

2) $\vec{E}(t) = Re\left[\vec{E}_\omega e^{i(\delta-\omega t)}\right]$ where $\vec{E}_\omega$ is real

a) induced dipole moment is

$$\vec{p}(t) = q\vec{r}(t) \qquad \vec{r}(t) = Re\left[\vec{r}_\omega e^{i(\delta-\omega t)}\right]$$

$$= Re\left[q\vec{r}_\omega e^{i(\delta-\omega t)}\right]$$

$$= Re\left[\vec{p}_\omega e^{i(\delta-\omega t)}\right]$$

$$\vec{p}_\omega = q\vec{r}_\omega = \alpha(\omega)\vec{E}_\omega$$ by definition of $\alpha(\omega)$

$$\vec{r}_\omega = \frac{\alpha(\omega)\vec{E}_\omega}{q}$$

velocity $$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt} Re\left[\vec{r}_\omega e^{i(\delta-\omega t)}\right]$$

$$= Re\left[-i\omega\vec{r}_\omega e^{i(\delta-\omega t)}\right]$$

$$\Rightarrow \quad \vec{v}_\omega = -i\omega\vec{r}_\omega$$

$$\boxed{\vec{v}_\omega = \frac{-i\omega\alpha(\omega)}{q}\vec{E}_\omega}$$

b) work done in time dt is

$$dW = \vec{F}\cdot\vec{v}\,dt = q\vec{E}(t)\cdot\vec{v}(t)\,dt$$

$$\vec{E}(t) = \mathrm{Re}\left[\vec{E}_\omega e^{i(\delta-\omega t)}\right] = \vec{E}_\omega \cos(\delta-\omega t)$$

$$\vec{v}(t) = \mathrm{Re}\left[\vec{v}_\omega e^{i(\delta-\omega t)}\right] = \mathrm{Re}\left[-\frac{i\omega(\alpha_1+i\alpha_2)}{q}\vec{E}_\omega e^{i(\delta-\omega t)}\right]$$

$$= \frac{\omega\alpha_1}{q}\vec{E}_\omega \sin(\delta-\omega t) + \frac{\omega\alpha_2}{q}\vec{E}_\omega \cos(\delta-\omega t)$$

$$\frac{W}{T} = \frac{1}{T}\int_0^T dt\; q\vec{E}(t)\cdot\vec{v}(t)$$

$$= \frac{1}{T}\int_0^T dt\left[\omega\alpha_1\cos(\delta-\omega t)\sin(\delta-\omega t) + \omega\alpha_2\cos^2(\delta-\omega t)\right]E_\omega^2$$

$$= \left[\omega\alpha_1 \underbrace{\langle\cos(\delta-\omega t)\sin(\delta-\omega t)\rangle}_{\text{average over one period is zero}} + \omega\alpha_2 \underbrace{\langle\cos^2(\delta-\omega t)\rangle}_{\text{average over one periode is } 1/2}\right]E_\omega^2$$

$$\boxed{\frac{W}{T} = \frac{1}{2}\omega\alpha_2 E_\omega^2}$$

work done depends on only the imaginary part of $\alpha(\omega)$

c) wave is $\vec{E}(\vec{r},t) = \vec{E}_\omega e^{-k_2 z} \cos(k_1 z - \omega t)$

use result of part (b) with substitution $\begin{cases} \vec{E}_\omega \to \vec{E}_\omega e^{-k_2 z} \\ \delta \to k_1 z \end{cases}$

to get the work per period of oscillation done on a slab of thickness dz at depth z into the medium

$$\underbrace{\left(\tfrac{1}{2}\omega\alpha_2 \,|\vec{E}_\omega e^{-k_2 z}|^2\right)}_{\text{work per atom per period}} \underbrace{(N A\, dz)}_{\#\text{ atoms in slab}}$$

Total work, per area, per period on the half space $z > 0$ is then

$$\frac{W}{TA} = \frac{1}{2}\omega\alpha_2 E_\omega^2 N \int_0^\infty dz\, e^{-2k_2 z}$$

$$= \frac{1}{2}\omega\alpha_2 E_\omega^2 N \frac{1}{2k_2}$$

to express the answer in terms of ε as the problem asks,

use $\varepsilon = 1 + 4\pi N\alpha \Rightarrow \varepsilon_2 = 4\pi N\alpha_2$ so $N\alpha_2 = \frac{\varepsilon_2}{4\pi}$

$$\boxed{\frac{W}{TA} = \frac{1}{16\pi}\frac{\omega}{k_2}\varepsilon_2 E_\omega^2}$$

work per period depends only on the imaginary part of $\varepsilon(\omega)$.

d) $\vec{B}(\vec{r},t) = \dfrac{c|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\,e^{-k_2 z}\cos(k_1 z-\omega t+\varphi)$

poynting vector

$$\vec{S}(\vec{r},t) = \frac{c}{4\pi}\,\vec{E}\times\vec{B}$$

$$= \frac{c^2|k|}{4\pi\omega}\left[\vec{E}_\omega\times(\hat{z}\times\vec{E}_\omega)\right]e^{-2k_2 z}\cos(\Phi)\cos(\Phi+\varphi)$$

where $\Phi = k_1 z-\omega t$

$$= \frac{c^2|k|}{4\pi\omega}E_\omega^2\,e^{-2k_2 z}\cos(\Phi)\cos(\Phi+\varphi)\,\hat{z}$$

The average over one period of oscillation is then

$$\langle\vec{S}(\vec{r})\rangle = \frac{c^2|k|}{4\pi\omega}E_\omega^2\,e^{-2k_2 z}\left\langle\cos\Phi\cos(\Phi+\varphi)\right\rangle$$

where $\langle\cdots\rangle$ is $\frac{1}{T}\int_0^T dt\,(\cdots)$ the average over one period

$$\cos(\Phi+\varphi) = \cos\Phi\cos\varphi - \sin\Phi\sin\varphi$$

So

$$\langle\cos\Phi\cos(\Phi+\varphi)\rangle = \underbrace{\langle\cos^2\Phi\rangle}_{=1/2}\cos\varphi - \underbrace{\langle\cos\Phi\sin\Phi\rangle}_{=0}\sin\varphi$$

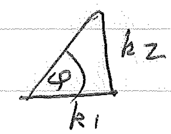
$$= \frac{1}{2}\cos\varphi$$

$$\langle\vec{S}(\vec{r})\rangle = \frac{c^2|k|}{8\pi\omega}E_\omega^2\,e^{-2k_2 z}\cos\varphi\,\hat{z}$$

so the energy per period, per unit area, flowing through the plane at $z=0$ is just

$$\langle \vec{S}(z=0) \rangle \cdot \hat{z} = \frac{c^2 |k|}{8\pi\omega} E_\omega^2 \cos\varphi$$

now $\tan\varphi = k_2/k_1$



so

$$\cos\varphi = \frac{k_1}{\sqrt{k_1^2+k_2^2}} = \frac{k_1}{|k|}$$

So

$$\langle \vec{S} \rangle \cdot \hat{z} = \frac{c^2 k_1}{8\pi\omega} E_\omega^2$$

To show that this is the same result as in part (c) note

$$k^2 = (k_1 + i k_2)^2 = k_1^2 - k_2^2 + 2 i k_1 k_2 = \frac{\omega^2}{c^2}(\varepsilon_1 + i \varepsilon_2)$$

$$\Rightarrow \quad k_1 k_2 = \frac{\omega^2}{2c^2}\varepsilon_2 \qquad \text{so} \qquad c^2 k_1 k_2 = \omega^2 \varepsilon_2$$

So

$$\langle \vec{S} \rangle \cdot \hat{z} = \frac{c^2 k_1}{8\pi\omega} E_\omega^2 \left(\frac{k_2}{k_2}\right) = \frac{1}{16\pi} \frac{\omega^2 \varepsilon_2}{\omega k_2} E_\omega^2$$

$$\boxed{\langle \vec{S} \rangle \cdot \hat{z} = \frac{1}{16\pi} \frac{\omega}{k_2} \varepsilon_2 E_\omega^2}$$

same result as in part (c)

So the energy flowing into the half space $z > 0$ all gets dissipated in the work done on oscillating the polarized electrons. This is why the wave amplitude $e^{-k_2 z} \to 0$ as $z \to \infty$.

e) For a region of total reflection we had

$$\varepsilon_1 < 0 , \quad \varepsilon_2 \approx 0$$

Since the total work done on the material is proportional to ε_2 then the total work done is ≈ 0.