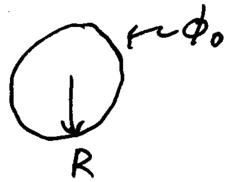
PHY 415 Midterm Solutions

1) a)



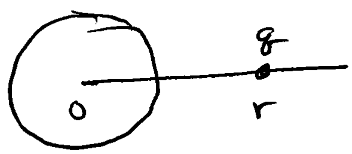
solution for potential $\phi(r)$ such that $\phi(R) = \phi_0$ is

$$\phi(r) = \frac{\phi_0 R}{r}$$

This is same as coulomb potential for a charge

$$\boxed{Q = \phi_0 R}$$

b)



point charge q would induce a charge $q' = -q\frac{R}{r}$ if the sphere were grounded, ie $\phi_0 = 0$. To make the potential equal ϕ_0 on the surface we need to put an extra image charge $\phi_0 R$ at origin. Therefore the total charge induced on the sphere is

$$\boxed{Q = \phi_0 R - q\frac{R}{r}}$$

c) Force of attraction is between q and the image charge $-q\frac{R}{r}$ located at $a = \frac{R^2}{r}$ and the charge $\phi_0 R$ located at the origin

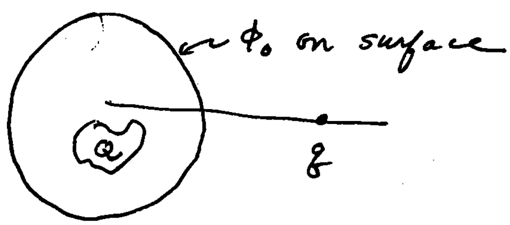
$$F = \frac{q\phi_0 R}{r^2} - \frac{q^2 R}{r}\frac{1}{(r - R^2/r)^2}$$

$$\boxed{F = qR\left[\frac{\phi_0}{r^2} - \frac{qr}{(r^2-R^2)^2}\right]}$$

F is always attractive if one is sufficiently close to the surface of the sphere.

But if $q\phi_0 > 0$, then F will be repulsive when q is sufficiently far from the surface

d)

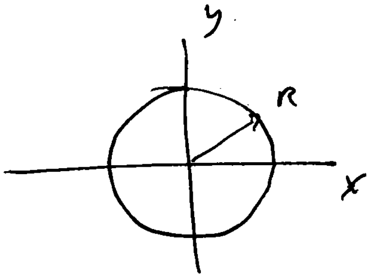


We know that there is a unique solution for ϕ outside the sphere, for a given charge distribution outside and specified ϕ on the surface.

Since these remain the same as in part (b), the ϕ outside is not effected at all by the presence of the charge in the cavity.

$\Rightarrow$ force is the same as in part (c)!

2)



$$\sigma(r,\varphi) = Ar \sin 2\varphi$$
$$= 2Ar \sin\varphi \cos\varphi$$

monopole $\quad q = \int_0^R dr\, r \int_0^{2\pi} d\varphi\ \sigma(r,\varphi)$

$$= \int_0^R dr\ 2Ar^2 \int_0^{2\pi} d\varphi\ \sin 2\varphi = 0$$

dipole $\quad \vec{p} = \int_0^R dr\ 2Ar^2 \int_0^{2\pi} d\varphi\ \sin\varphi \cos\varphi \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix}$

↑ x, y, z, components

$= 0$ since the angular integrals vanish

leading term is electric quadrupole

$$\overleftrightarrow{Q} = \int d^3r\ \rho(\vec{r}) \left[3\vec{r}\vec{r} - r^2 \overleftrightarrow{I} \right]$$

since $\rho(\vec{r}) = 0$ except on the disk where $z = 0$
$\overleftrightarrow{Q}$ has the form:

$$\begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{xy} & Q_{yy} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}$$

evaluate these one by one.
Note that the piece in $\overleftrightarrow{Q}$ that is $-r^2\overleftrightarrow{I}$ always vanishes

$$\int d^3r\ \rho(\vec{r})\, r^2\, \overleftrightarrow{I} = A\overleftrightarrow{I} \int_0^R dr\, r^3 \int_0^{2\pi} d\varphi\ \sin 2\varphi = 0$$

angular integration vanishes

$\Rightarrow Q_{zz} = 0$

$$Q_{xx} = \int_0^R dr\, r \int_0^{2\pi} d\varphi\; 2Ar\sin\varphi\cos\varphi\, 3\underbrace{r^2\cos^2\varphi}_{x^2}$$

$$= 6A\int_0^R dr\, r^4 \int_0^{2\pi} d\varphi\, \sin\varphi\cos^3\varphi$$

$$= \frac{6AR^5}{5}\left(-\frac{\cos^4\varphi}{4}\right)_0^{2\pi} = 0$$

$$Q_{yy} = \int_0^R dr\, r \int_0^{2\pi} d\varphi\; 2Ar\sin\varphi\cos\varphi\, 3\underbrace{r^2\sin^2\varphi}_{y}$$

$$= \frac{6AR^5}{5}\left(\frac{\sin^4\varphi}{4}\right)_0^{2\pi} = 0$$

$$Q_{xy} = Q_{yx} = \int_0^R dr\, r \int_0^{4\pi} d\varphi\; 2Ar\sin\varphi\cos\varphi\; 3\underbrace{r^2\sin\varphi\cos\varphi}_{xy}$$

$$= \frac{6AR^5}{5}\int_0^{2\pi} d\varphi\,(\sin\varphi\cos\varphi)^2$$

$$= \frac{6AR^5}{5}\,\frac{1}{4}\int_0^{2\pi} d\varphi\,(\sin 2\varphi)^2$$

$$= \frac{6AR^5}{5}\,\frac{1}{4}\,\pi = \frac{3\pi}{10}AR^5$$

$$\overleftrightarrow{Q} = \frac{3\pi}{10}AR^5\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

potential in quadrapole approx is

$$\phi \simeq \frac{1}{2} \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{r^3}$$

where in spherical coords $\hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

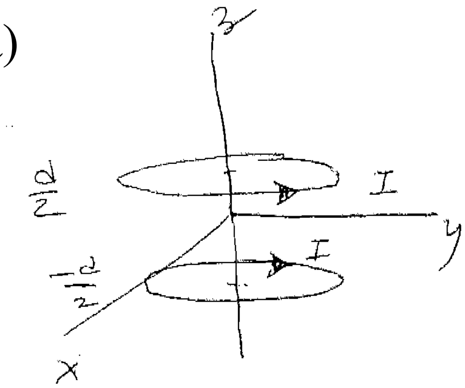
$$\phi = \frac{3\pi A R^5}{20 r^3} \hat{r} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$= \frac{3\pi A R^5}{20 r^3} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \begin{pmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

$$= \frac{3\pi A R^5}{20 r^3} \left[\sin^2\theta\cos\varphi\sin\varphi + \sin^2\theta\sin\varphi\cos\varphi \right]$$

$$\boxed{\phi = \frac{3\pi A R^5}{20 r^3} \sin^2\theta \sin 2\varphi}$$

3) a)



dipole moment of loop at $z = \frac{d}{2}$

is $\vec{m}_1 = \frac{\pi a^2}{c} I \, \hat{z}$

dipole moment of loop at $z = -\frac{d}{2}$

is $\vec{m}_2 = -\frac{\pi a^2}{c} I \, \hat{z}$

since current is clockwise

total magnetic dipole moment is

$$\vec{m} = \vec{m}_1 + \vec{m}_2 = 0$$

Above configuration is a <u>magnetic quadrapole</u>!

So we expect that at $r \gg a$ the magnetic field goes as $\boxed{\vec{B} \sim 1/r^4}$

3) b)

To get information about the angular dependence of $\vec{B}$ we need to do a calculation, We can use the magnetic scalar potential Φ_M, the same way as we did a single loop in class.

First we find the exact solution for $\vec{B}$ on the z axis.

We have $\vec{B} = -\vec{\nabla}\phi_M$. Because of the rotational symmetry about the $\hat{z}$ axis we can then write

$$\phi_M(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

there are no $A_\ell r^\ell$ terms since we want $\phi_M \to 0$ as $r \to \infty$

Then

$$\vec{B} = -\vec{\nabla}\phi_M = -\frac{\partial\phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi_M}{\partial\theta}\hat{\theta}$$

$$= \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\hat{r} - \frac{B_\ell}{r^{\ell+2}}\frac{\partial P_\ell(\cos\theta)}{\partial\theta}\hat{\theta}\right]$$

use $\frac{\partial P_\ell}{\partial\theta} = \frac{\partial P_\ell}{\partial x}\frac{dx}{d\theta} = -P_\ell' \sin\theta$ where $x = \cos\theta$, $P_\ell' = \frac{dP_\ell}{dx}$

So

$$\vec{B}(\vec{r}) = \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\hat{r} + \frac{B_\ell}{r^{\ell+2}} \sin\theta P_\ell'(\cos\theta)\hat{\theta}\right]$$

We now need to determine the unknown coefficients B_ℓ. To do that, we consider $\vec{B}$ for $\vec{r} = z\hat{z}$ along the $+z$ axis, and compare to the exact solution.

For $\theta = 0$, $\cos\theta = 1$, $\sin\theta = 0$, $r = z$, and we have

$$\boxed{\vec{B}(z\hat{z}) = \sum_{\ell=0}^{\infty} \frac{(\ell+1)B_\ell}{z^{\ell+2}}\hat{z}}$$

where we used $P_\ell(1) = 1$

Now we need to compare the above to the exact solution along the z axis.

For a single loop at $z=0$ we have, as given,

$$\vec{B}(z\hat{z}) = \frac{2\pi a^2 I}{c z^3} \frac{1}{\left[1+\left(\frac{a}{z}\right)^2\right]^{3/2}} \hat{z} = \frac{2\pi a^2 I}{c} \frac{1}{\left[z^2+a^2\right]^{3/2}} \hat{z}$$

For the loop at $z=\frac{d}{2}$, we use the above formula, but replace $z \to z-\frac{d}{2}$

For the loop at $z=-\frac{d}{2}$, we use the above formula, but take $z \to z+\frac{d}{2}$.

We then get

since I is clockwise in lower loop

$$\vec{B}(z\hat{z}) = \frac{2\pi a^2 I}{c}\left\{\frac{1}{\left[(z-\frac{d}{2})^2+a^2\right]^{3/2}} - \frac{1}{\left[(z+\frac{d}{2})^2+a^2\right]^{3/2}}\right\}\hat{z}$$

$$= \frac{2\pi a^2 I}{c z^3}\left\{\frac{1}{\left[(1-\frac{d}{2z})^2+(\frac{a}{z})^2\right]^{3/2}} - \frac{1}{\left[(1+\frac{d}{2z})^2+(\frac{a}{z})^2\right]^{3/2}}\right\}\hat{z}$$

$$= \frac{2\pi a^2 I}{c z^3}\left\{\frac{1}{\left[1-\frac{d}{2}+(\frac{d}{z})^2+(\frac{a}{z})^2\right]^{3/2}} - \frac{1}{\left[1+\frac{d}{2}+(\frac{d}{z})^2+(\frac{a}{z})^2\right]^{3/2}}\right\}$$

for $z \gg a$, and $a \approx d$, we can expand to lowest order in the small quantities $\left(\frac{d}{z}\right)$ and $\left(\frac{a}{z}\right)$ to get

$$\frac{1}{\left[1-\frac{d}{2}+(\frac{d}{z})^2+(\frac{a}{z})^2\right]^{3/2}} \simeq 1+\frac{3}{2}\frac{d}{z}$$

lowest order small term

higher order small terms, can ignore

Similarly, to lowest order in the small quantities

$$\frac{1}{\left[1+\frac{d}{z}+\left(\frac{d}{z}\right)^2+\left(\frac{a}{z}\right)^2\right]^{3/2}} \simeq 1-\frac{3}{2}\frac{d}{z}$$

Putting the two pieces together we get

$$\vec{B}(z\hat{z}) = \frac{2\pi a^2 I}{c z^3}\left[\left(1+\frac{3}{2}\frac{d}{z}\right)-\left(1-\frac{3}{2}\frac{d}{z}\right)\right]\hat{z}$$

$$\boxed{\vec{B}(z\hat{z}) = \frac{6\pi a^2 I d}{c z^4}\hat{z}} + \text{higher order terms } \frac{1}{z^n},\ n \geq 5$$

From the above, and comparing to our Legendre polynomial expansion from $-\vec{\nabla}\Phi_M$

$$\vec{B}(z\hat{z}) = \sum_{l=0}^{\infty} \frac{(l+1)B_l}{z^{l+2}}\hat{z}$$

we see that the leading term is the $l=2$ magnetic quadrupole term, and

$$\boxed{B_2 = \frac{2\pi a^2 I d}{c}}$$

We then have for arbitrary θ, to leading order,

$$\vec{B}(\vec{r}) = \frac{2\pi a^2 I d}{c}\left[3P_2(\cos\theta)\hat{r} + \sin\theta\, P_2'(\cos(\theta))\hat{\theta}\right]$$

$P_2(x) = \frac{1}{2}(3x^2-1)$ so $P_2'(x) = 3x$

$$\boxed{\vec{B}(\vec{r}) = \frac{2\pi a^2 I d}{c}\left[-\frac{3}{2}(3\cos^2\theta-1)\,\hat{r} + -3\sin\theta\cos\theta\,\hat{\theta}\right]}$$

We can sketch the field lines for this magnetic quadrupole as below