PHYS 415 Solutions Endterm Exam 2024

1) a) That $\varepsilon(\omega)$ varies with ω results in:

i) $\vec{E}(t)$ and $\vec{D}(t)$ (or equivalently $\vec{P}(t)$) are non-locally related in time.

ii) waves of different ω travel with different speed

iii) wave pulses travel with group velocity $v_g = \frac{d\omega}{dk}$ and not the phase velocity $v_p = \omega/k$

iv) waves can have different behavior depending on the value of ω, ie we can have transparent propagation, resonant absorption, total reflection.

That $\varepsilon(\omega)$ may have an imaginary component

i) waves decay as they propagate $e^{-k_2 z}$

ii) $\vec{E}$ and $\vec{B}$ oscillations are shifted in phase with respect to one another with phase shift $\delta = \arctan(k_2/k_1)$

b) At sufficiently low frequency, the wave in the dielectric has transparent propagation, while the wave in the conductor is strongly damped ("good" conductor has large $\varepsilon_2 \gg \varepsilon_1$)

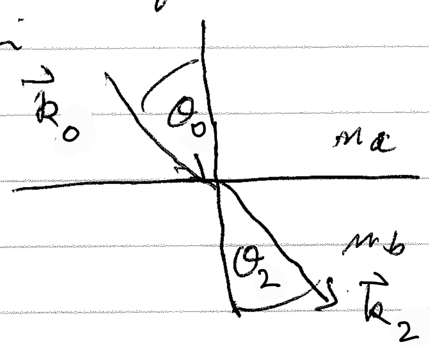
Also, wave in dielectric at low ω has $\vec{E}$ and $\vec{B}$ oscillating in phase, while in the conductor at low ω, $\vec{E}$ and $\vec{B}$ oscillate $\sim 45°$ out of phase (since $k_1 \approx k_2$)

c) Plasma freq in conductor ω_p

i) At large ω, ω_p is the threshold between total reflection at $\omega < \omega_p$ and transparent propagation at $\omega > \omega_p$

ii) There can be longitudinal oscillations of the electric field $(\vec{E} \parallel \vec{k})$ and oscillations of charge density only at freq ω_p

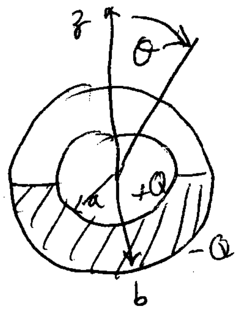
d) Snell's law relates the angle of incidence to the angle of transmission for an EM wave hitting a planar interface $n_a \sin\theta_0 = n_b \sin\theta_2$ where θ_0 and θ_2 are the angles of incidence and transmission, and n_a and n_b are the indices of refraction of the incident medium and the transmitted medium



Snell's law is not valid if medium b is strongly dissipative. Snell's law holds only when both media a and b are nearly transparent

e) For a wave hitting an interface at an angle of incidence equal to Brewster's angle $\theta_0 = \theta_B$, the reflected wave will be polarized with its electric field perpendicular to the plane of incidence (since reflection coefficient of parallel polarization is $R_{\parallel} = 0$)

2)



Assume solution is a radial electric field

$\vec{E}(\vec{r}) = E(r)\,\hat{r}$ same for all θ

$\Rightarrow$ total charge density σ_{tot} is uniform on the inner sphere, and $-\sigma_{tot}$ uniform on outer sphere

$$\Rightarrow E(r) = \frac{4\pi a^2 \sigma_{tot}}{r^2} \qquad a < r < b$$

at $r = a$
$$\sigma_{tot} = \begin{cases} \sigma_{free} & 0 < \theta < \frac{\pi}{2} \text{ as no dielectric} \\ \sigma'_{free} + \sigma_b & \frac{\pi}{2} < \theta < \pi \text{ as } \sigma_b \text{ on dielectric} \end{cases}$$

where we know $2\pi a^2(\sigma_{free} + \sigma'_{free}) = Q$ total free charge on inner surface

and for a linear dielectric, free charge is screened by ϵ, so

$$\sigma_{tot} = \frac{\sigma'_{free}}{\epsilon}$$

so

$$2\pi a^2(\sigma_{free} + \sigma'_{free}) = 2\pi a^2(\sigma_{tot} + \epsilon\,\sigma_{tot}) = 2\pi a^2(1+\epsilon)\,\sigma_{tot}$$
$$= Q$$

$$\Rightarrow \boxed{\sigma_{tot} = \frac{Q}{2\pi a^2(1+\epsilon)}}$$

a) $\boxed{\vec{E} = \frac{2Q}{(1+\epsilon) r^2} \hat{r}}$ $\quad a < r < b$

b) $\boxed{\sigma_{tot} = \frac{Q}{2\pi a^2 (1+\epsilon)}}$ $\quad r = a$

c) $\sigma_b = \sigma_{tot} - \sigma'_{free} = \sigma_{tot} - \epsilon \sigma_{tot} = (1-\epsilon)\sigma_{tot}$

$$\boxed{\sigma_b = \frac{(1-\epsilon) Q}{2\pi a^2 (1+\epsilon)}}$$

above gives a self consistent solution therefore it is the unique solution!

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to prove that $\vec{E}(\vec{r}) = E(r)\hat{r} = \frac{Q_{tot}}{r^2}\hat{r}$

note that the total charge in the region $a < r < b$ vanishes since there is no free charge there and $\rho_{tot} = \frac{\rho_{free}}{\epsilon}$

$\Rightarrow \nabla^2 \phi = 0$ in this region where ϕ is the usual electrostatic potential, $\vec{E} = -\vec{\nabla}\phi$

Azimuthal symmetry $\Rightarrow \phi(r,\theta) = \sum_\ell \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$

At $r = a$, ϕ is a constant, since surface of sphere is equipotential

$$\Rightarrow \phi(a,\theta) = \sum_\ell \left[A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} \right] P_\ell(\cos\theta)$$

must be independent of θ.

$\Rightarrow$ coefficients of P_ℓ must vanish for $\ell \neq 0$

$$\Rightarrow \quad A_\ell = \frac{-B_\ell}{a^{2\ell+1}}$$

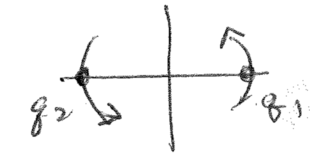
but same arguement holds at $r = b$

$$\Rightarrow A_\ell = -\frac{B_\ell}{b^{2\ell+1}}$$

only way to solve both conditions is $A_\ell = B_\ell = 0$ for $\ell \neq 0$

$$\Rightarrow \phi(r,\theta) = \left[A_0 + \frac{B_0}{r}\right] \qquad \text{since } P_0(\cos\theta) = 1$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi = \frac{B_0}{r^2}\,\hat{r}$$

3) 

position of q_1 is $\vec{r}_0(t)$

position of q_2 is $-\vec{r}_0(t)$

a) electric charge density is $\qquad \vec{r}_0(t) = d(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

$$\rho(\vec{r},t) = q_1\,\delta(\vec{r}-\vec{r}_0(t)) + q_2\,\delta(\vec{r}+\vec{r}_0(t))$$

$$\vec{p}(t) = q_1\,\vec{r}_0(t) - q_2\,\vec{r}_0(t) = (q_1 - q_2)\,\vec{r}_0(t)$$

for $q_1 = -q_2 = q$

$$\vec{p}(t) = 2q\,d(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$$

$$= Re\left\{ 2qd\left(e^{-i\omega t}\,\hat{x} + i e^{-i\omega t}\,\hat{y}\right)\right\}$$

$$= Re\left\{ 2qd\,(\hat{x}+i\hat{y})\,e^{-i\omega t}\right\}$$

$$\vec{p}_\omega = 2qd\,(\hat{x}+i\hat{y})$$

$$\vec{E} = Re\left\{ k^2\,\frac{e^{i(kr-\omega t)}}{r}\,\hat{r}\times(\vec{p}_\omega\times\hat{r})\right\}$$

$$= \frac{2qd\,k^2}{r}\,Re\left\{ e^{i(kr-\omega t)}\,\hat{r}\times\left((\hat{x}+i\hat{y})\times\hat{r}\right)\right\}$$

$$\boxed{\vec{E} = \frac{2qd\,k^2}{r}\left\{\cos(kr-\omega t)\left[\hat{r}\times(\hat{x}\times\hat{r})\right] - \sin(kr-\omega t)\left[\hat{r}\times(\hat{y}\times\hat{r})\right]\right.}$$

$$\vec{B} = Re\left\{ -k^2\,\frac{e^{i(kr-\omega t)}}{r}\,\vec{p}_\omega\times\hat{r}\right\}$$

$$= -\frac{2qd\,k^2}{r}\,Re\left\{ e^{i(kr-\omega t)}\,(\hat{x}+i\hat{y})\times\hat{r}\right\}$$

$$\boxed{\vec{B} = -\frac{2qd\,k^2}{r}\left\{\cos(kr-\omega t)\left[\hat{x}\times\hat{r}\right] - \sin(kr-\omega t)\left[\hat{y}\times\hat{r}\right]\right\}}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

$$= -\frac{4q^2d^2k^4c}{4\pi r^2}\Big\{\cos^2(kr-\omega t)\,[\hat{r}\times(\hat{x}\times\hat{r})]\times[\hat{x}\times\hat{r}]$$

$$+\sin^2(kr-\omega t)\,[\hat{r}\times(\hat{y}\times\hat{r})]\times[\hat{y}\times\hat{r}]$$

$$-\cos(kr-\omega t)\sin(kr-\omega t)\Big([\hat{r}\times(\hat{x}\times\hat{r})]\times[\hat{y}\times\hat{r}]$$

$$+[\hat{r}\times(\hat{y}\times\hat{r})]\times[\hat{x}\times\hat{r}]\Big)\Big\}$$

average: $\cos^2 \to \frac{1}{2}$, $\sin^2 \to 1/2$, $\cos\sin \to 0$

$$\langle\vec{S}\rangle = -\frac{2q^2d^2k^4c}{4\pi r^2}\Big\{[\hat{r}\times(\hat{x}\times\hat{r})]\times[\hat{x}\times\hat{r}] + [\hat{r}\times(\hat{y}\times\hat{r})]\times[\hat{y}\times\hat{r}]\Big\}$$

$$= \frac{2q^2d^2k^4c}{4\pi r^2}\Big\{\hat{r}\,(\hat{x}\times\hat{r})\cdot(\hat{x}\times\hat{r}) - (\hat{x}\times\hat{r})\,\hat{r}\cdot(\hat{x}\times\hat{r})$$

$$+\hat{r}\,(\hat{y}\times\hat{r})\cdot(\hat{y}\times\hat{r}) - (\hat{y}\times\hat{r})\,\hat{r}\cdot(\hat{y}\times\hat{r})\Big\}$$

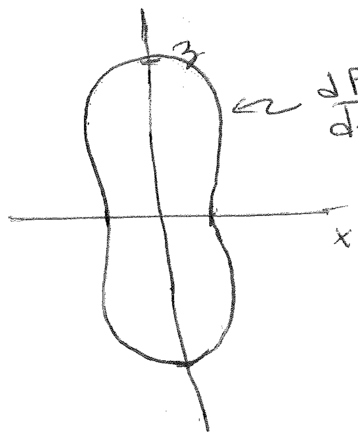
$$= \frac{2q^2d^2k^4c}{4\pi r^2}\,\hat{r}\,\Big\{(\hat{x}\times\hat{r})\cdot(\hat{x}\times\hat{r}) + (\hat{y}\times\hat{r})(\hat{y}\times\hat{r})\Big\}$$

$$= \frac{2q^2d^2k^4c}{4\pi r^2}\,\hat{r}\,\Big\{1-(\hat{x}\cdot\hat{r})^2 + 1 - (\hat{y}\cdot\hat{r})^2\Big\}$$

$$= \frac{2q^2d^2k^4c}{4\pi r^2}\,\hat{r}\,\Big\{2 - \sin\theta^2\cos^2\phi - \sin^2\theta\sin^2\phi\Big\}$$

$$\boxed{\langle\vec{S}\rangle = \frac{4q^2d^2k^4c}{4\pi r^2}\,\hat{r}\,\Big\{1-\frac{1}{2}\sin^2\theta\Big\}}$$

$$\frac{dP}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r} = \frac{4 q^2 d^2 k^4 c}{4\pi} \left(1 - \frac{1}{2} \sin^2 \theta \right)$$



$\frac{dP}{d\Omega}$ polar plot.