Unit 1-5: Review of Fourier Transforms

For a function $f(\mathbf{r})$, the Fourier transform and its inverse is given by

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} d^3r\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) \qquad \text{Fourier transform} \tag{1.5.1}$$

$$f(\mathbf{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}) \qquad \text{inverse transform} \tag{1.5.2}$$

In the above, we denoted the Fourier transform of f by $\tilde{f}$. Later, we will dispense with that notation and you will know whether we are talking about the function or its transform by the argument of the function, i.e., $f(\mathbf{r})$ is the function while $f(\mathbf{k})$ is the transform. Note, different texts sometime use different notations. Sometimes the transform is defined with a + sign in the exponent, while the inverse transform has the − sign. Sometimes the factor $1/(2\pi)^3$ is put in the transform instead of the inverse transform. In quantum mechanics, one usually defines both the transform and the inverse to have a factor $1/\sqrt{(2\pi)^3}$. So just be sure when you are reading a text or an article that you understand what convention the author is using to define the transforms.

Some special cases well worth remembering

1) The transform of the Dirac delta function is

$$\int d^3r\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \delta(\mathbf{r}-\mathbf{r}_0) = \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_0} \tag{1.5.3}$$

The inverse is then

$$\delta(\mathbf{r}-\mathbf{r}_0) = \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_0} \qquad \Rightarrow \qquad \delta(\mathbf{r}-\mathbf{r}_0) = \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} \tag{1.5.4}$$

or letting $\mathbf{r} \leftrightarrow \mathbf{k}$ in the above

$$\delta(\mathbf{k}-\mathbf{k}_0) = \int \frac{d^3r}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}_0)} \tag{1.5.5}$$

2) The transform of the Coulomb potential $\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

We know that

$$\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{1.5.6}$$

Let

$$f(\mathbf{k}) \equiv \int d^3r\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{be the Fourier transform of } \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.5.7}$$

Substitute

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{k}) \qquad \text{and} \qquad \delta(\mathbf{r}-\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{1.5.8}$$

into the Poisson's equation (1.5.6) to get

$$\nabla^2 \left[\int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{k}) \right] = -4\pi \int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{1.5.9}$$

For the term on the left hand side, the operator ∇^2 acts only on the variable $\mathbf{r}$, so we can move it inside the integral and let it act on the exponential term $e^{i\mathbf{k}\cdot\mathbf{r}}$.

$$\nabla^2 \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \right) \tag{1.5.10}$$

To evaluate we have

$$\boldsymbol{\nabla}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{i=1}^{3} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{i=1}^{3} \hat{\mathbf{x}}_i \, ik_i \, e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.5.11}$$

where x_1, x_2, x_3 correspond to x, y, z.

Next,

$$\boldsymbol{\nabla}\cdot\left(i\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\right) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} ik_i\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{i=1}^{3}(ik_i)(ik_i)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = (i\mathbf{k})\cdot(i\mathbf{k})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = -k^2\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.5.12}$$

So

$$\nabla^2\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = -k^2\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.5.13}$$

The Poisson's equation (1.5.9) then becomes

$$\int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}(-k^2)f(\mathbf{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}'} \tag{1.5.14}$$

or

$$\int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\left[-k^2f(\mathbf{k})\right] = \int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\left[-4\pi\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}'}\right] \tag{1.5.15}$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal. Equating the terms in the square brackets above we get

$$-k^2f(\mathbf{k}) = -4\pi\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}'} \quad \Rightarrow \quad \boxed{f(\mathbf{k}) = \frac{4\pi}{k^2}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}'}} \quad \text{is the Fourier transform of } \frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{1.5.16}$$