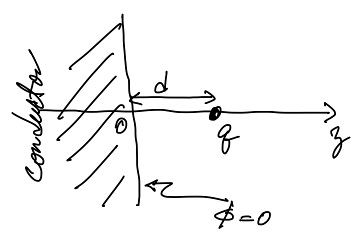
Unit 2-2: The Image Charge Method

We now discuss several different method for solving Poisson's equation for the potential ϕ in simple geometries. The first method is the image charge method.

For simple geometries, we can try to obtain the Green's functions G_D or G_N (for Dirichlet and Neumann boundary condition) by placing a set of *image charges* outside the volume V of interest – i.e. on the "other side" of the system boundary surface S. Because these image charges are outside V, their contribution to the potential inside V obeys $\nabla^2\phi^{\text{image}} = 0$. One then chooses the location of the image charges so that the total ϕ (from both the real charges in V and the image charges outside V) will obey the desired boundary condition.

1) The first case we consider is a charge q placed a distance d along the $\hat{\mathbf{z}}$ axis in front of an infinite flat grounded plane ($\phi = 0$ on the plane).



We want $\nabla^2\phi = -4\pi q\delta(x)\delta(y)\delta(z-d)$, and $\phi = 0$ for $z = 0$. If we find a solution to the above problem, we know it is the unique solution.

Solution: put a fictitious image charge $-q$ at $z = -d$. The potential ϕ is then the Coulomb potential from the real charge $+q$ and the image charge $-q$.

$$\phi(\mathbf{r}) = \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y + 2 + (z+d)^2}} \tag{2.2.1}$$

The first term is from the real charge $+q$, the second term is from the image charge $-q$. Let us check that this ϕ is indeed the solution to our problem.

Firstly, $\nabla^2\phi(\mathbf{r}) = -4\pi q\delta(\mathbf{r} - d\hat{\mathbf{z}}) - 4\pi(-q)\delta(\mathbf{r} + \hat{\mathbf{z}}) = -4\pi q\delta(\mathbf{r} - d\hat{\mathbf{z}})$ for $\mathbf{r}$ in the volume V to the right of the grounded plane. The second delta function from the image charge is zero everywhere in V, because the zero of that delta function is at $\mathbf{r} = -z\hat{\mathbf{z}}$ which is *outside* V.

Secondly, we check the boundary condition.

$$\phi(x, y, z = 0) = \frac{q}{\sqrt{x^2 + y^2 + (-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (+d)^2}} = 0 \tag{2.2.2}$$

So $\phi = 0$ on the grounded plane as desired. Thus this ϕ is indeed the solution.

From a formal point of view, the solution of Eq. (2.2.1) gives the solution for a point charge q in V, subject to the condition that $\phi = 0$ on the planar surface. So this ϕ/q just gives the Dirichlet Green's function for the problem of the grounded plane. If the charge is at $\mathbf{r}'$, then the potential at $\mathbf{r}$ is

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{q}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \tag{2.2.3}$$

where $F(\mathbf{r}, \mathbf{r}')$ must obey $\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0$ in V and be such that $G_D(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r}$ on the boundary surface S. The image charge method tells us that $F(\mathbf{r}, \mathbf{r}') = -q/|\mathbf{r} - \mathbf{r}' - 2d\hat{\mathbf{z}}|$ is just the Coulomb potential from the image charge.

We can now use our solution of Eq. (2.2.1) to compute some interesting physical behavior. First we find the electric field $\mathbf{E}$ for $z > 0$ in the volume V. Since $\mathbf{E} = -\boldsymbol{\nabla}\phi$, the z component of the electric field is given by taking the derivative of Eq. (2.2.1) with respect to z,

$$E_z = -\frac{\partial \phi}{\partial z} = q\left[\frac{\left(\frac{1}{2}\right)2(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{\left(\frac{1}{2}\right)2(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \tag{2.2.4}$$

$$= q\left[\frac{(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \tag{2.2.5}$$

We can use this result to compute the surface charge $\sigma(x,y)$ on the surface of the conducting plane. Recall, the potential at a charged surface must obey the condition

$$-\frac{\partial \phi^{\text{top}}}{\partial n} + \frac{\partial \phi^{\text{bottom}}}{\partial n} = 4\pi\sigma \tag{2.2.6}$$

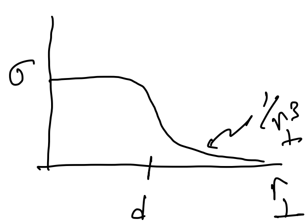
Here ϕ^{top} is the potential ϕ just to the right of the conductor surface, while ϕ^{bottom} is the potential just to the left of the conductor surface. Since ϕ is constant inside a conductor we have $\partial\phi^{\text{bottom}}/\partial n = 0$, and so,

$$-\frac{\partial \phi}{\partial n} = -\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = 4\pi\sigma \quad \Rightarrow \quad \sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial z} = \frac{1}{4\pi}E_z(x,y,z=0) \tag{2.2.7}$$

From Eq. (2.2.5) we have

$$\sigma = \frac{q}{4\pi}\left[\frac{-d}{\left[x^2+y^2+d^2\right]^{3/2}} - \frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}\right] = \frac{-qd}{2\pi\left[x^2+y^2+d^2\right]^{3/2}} = \frac{-qd}{2\pi\left[r_\perp^2+d^2\right]^{3/2}} \tag{2.2.8}$$

where $r_\perp = \sqrt{x^2+y^2}$.



Note, this form for $\sigma(r_\perp)$ seems reasonable – the surface charge σ is largest at $r_\perp = 0$ where the plane is closest to the real charge q, and then it decays as $r_\perp$ increases, and one moves further away from q. The length scale on which σ starts to decrease from its value at $r_\perp = 0$ is set by the distance d of the charge from the plane.

We can also ask, what is the total charge q^{induced} induced on the grounded plane by the presence of the charge q in front of the plane?

$$q^{\text{induced}} = \int_{-\infty}^{\infty} dxdy\,\sigma(x,y) = 2\pi\int_0^\infty dr_\perp r_\perp\,\sigma(r_\perp) = 2\pi\int_0^\infty dr_\perp r_\perp\,\frac{(-qd)}{2\pi[r_\perp^2+d^2]^{3/2}} \tag{2.2.9}$$

$$= -qd\left[\frac{-1}{]r_\perp^2+d^2]^{1/2}}\right]_0^\infty = -qd\left[0 - \frac{(-1)}{d}\right] = -q \tag{2.2.10}$$

So $q^{\text{induced}} = -q$, the total induced charge on the conducting surface is just equal to the image charge.

Finally, we can compute the force $\mathbf{F}$ on the charge q in front of the conducting plane. What is the origin of this force? It is the charge σ induced on the plane. What is the electric field produced by this σ? In the volume V to the right of the plane, the field from σ is exactly the same as the field that would be produced by the image charge $-q$.

$$\Rightarrow \qquad \mathbf{F} = \frac{-q\,q}{(2d)^2}\,\hat{\mathbf{z}} = \frac{-q^2}{4d^2}\,\hat{\mathbf{z}} \qquad F_z < 0 \;\Rightarrow\; \text{the force attracts the charge } q \text{ towards the plane} \tag{2.2.11}$$

What is the work that has to be done to move q into position from infinitely far away?

$$W = \int_\infty^d \boldsymbol{d\ell}\cdot(-\mathbf{F}) = -\int_\infty^d dz\,F_z = \int_d^\infty dz\,\left(\frac{-q^2}{4z^2}\right) = \frac{-q^2}{4d} \tag{2.2.12}$$

Why is the force used to compute W given by $-\mathbf{F}$? Why the minus sign? Because $\mathbf{F}$ is the electric force acting on q. If we want to move q around, we have to apply an equal but opposite force, $-\mathbf{F}$, to cancel out the electric force.

We see that $W < 0$, no matter what is the sign of the charge q. This is because the force $\mathbf{F}$ is attractive. Energy is released when the charge q moves towards the plane.

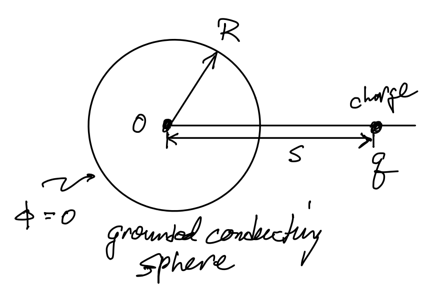
You might have thought that W would be equal to $q\phi^{\mathrm{image}} = -q^2/2d^2$, which is just the electrostatic energy of q in the potential of the image $-q$. But we see above that the correct W is one half of this. Why is that? When we say that the energy of a charge q in the presence of other charges is $q\phi$ (with ϕ being the electrostatic potential produced by the other charges), we get that result by moving q into position from infinity, *keeping the other charges fixed in place*. However, when we move the charge q in from infinity in the present example, the image charge $-q$ must also be moving, so as to aways stay equidistant from the plane on the opposite side.

Summary: What is going on in this problem? If there was no charge q, then the grounded conducting plane would be charge free with $\sigma = 0$. But now we put the charge q in front of the plane. If the plane stayed charge free, then the only electric field in the system would be that produced by q. But the field from q would not be perpendicular to the surface of the plane as it must be, because the plane is the surface of a conductor. The presence of the charge q in front of the plane therefore *induces* charge σ to appear on the plane. This σ arranges itself non-uniformly over the surface of the plane so that total electric field from q and from σ is then perpendicular to the plane. Where does this σ come from? It comes from the source that is causing the plane to stay grounded with $\phi = 0$. The image charge is a trick that lets us compute σ and hence the total $\mathbf{E}$ to the right of the plane where the charge q is. Note, the image charge must lie to the *left* of the plane, where we are *not* trying to find $\mathbf{E}$.

Discussion Question 2.2.1

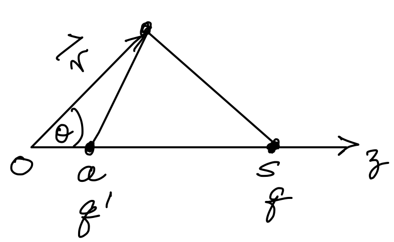
For the above problem, what is the total true physical electric field $\mathbf{E}$ to the *left* of the conducting plane? What is the $\mathbf{E}$ field on the left side of the plane that is produced by the induced charge σ? Suppose now that the conductor did *not* fill the half space $z < 0$, but was only a thin conducting plane of thickness w, with the right hand surface at $z = 0$, and the left hand surface at $z = -w$. What would the electric field be to the right, to the left, and inside the conducting plane?

2) Now consider a point charge q placed in front of a grounded ($\phi = 0$) conducting sphere of radius R. The charge is a distance $s > R$ from the center of the sphere.



We will try to solve this problem using the image charge method, placing an image charge q' on the *other side* of the bounding surface from where the real charge q is, i.e. by putting the image charge *inside* the sphere.

Where in the sphere should we put q'? Since our problem has rotational symmetry about the $\hat{\mathbf{z}}$ axis, we need to put q' on the $\hat{\mathbf{z}}$ axis. So we put q' at distance $a < R$, inside the sphere on the $\hat{\mathbf{z}}$ axis.



Now we compute the potential $\phi(\mathbf{r})$, at some position $\mathbf{r}$ outside the sphere, that arises from the real charge q and the image charge q'. Using the geometry shown in the sketch to the left, the position $\mathbf{r}$ is given by the radial distance r and the polar angle θ with respect to the origin. Then we have,

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r} - s\hat{\mathbf{z}}|} + \frac{q'}{|\mathbf{r} - a\hat{\mathbf{z}}|} = \frac{q}{\sqrt{r^2 + s^2 - 2sr\cos\theta}} + \frac{q'}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} \tag{2.2.13}$$

We now need to choose q' and a so that $\phi(R, \theta) = 0$ for all θ. Let us try to rewrite the denominators of the two terms above so that they look alike when $r = R$,

$$\phi(R,\theta) = \frac{q}{\sqrt{R^2+s^2-2sR\cos\theta}} + \frac{q'}{\sqrt{R^2+a^2-2aR\cos\theta}} \tag{2.2.14}$$

Write,

$$R^2+a^2-2aR\cos\theta = \frac{a}{s}\left(\frac{s}{a}R^2 + sa - 2sR\cos\theta\right) \tag{2.2.15}$$

so that the $\cos\theta$ terms in the two denominators will look alike. Then choose a so that $sa = R^2$, i.e. $\boxed{a = R^2/s}$, and so $sR^2/a = s^2$. Then the denominator of the second term is,

$$\left[\frac{R^2}{s^2}\left(s^2+R^2-2sR\cos\theta\right)\right]^{1/2} = \frac{R}{s}\left[s^2+R^2-2sR\cos\theta\right]^{1/2} \tag{2.2.16}$$

and so

$$\phi(R,\theta) = \frac{q}{[R^2+s^2-2sR\cos\theta]^{1/2}} + \frac{q'\left(\frac{s}{R}\right)}{[R^2+s^2-2sR\cos\theta]^{1/2}} \tag{2.2.17}$$

and the denominators of the two terms are now the same. So we can now make $\phi(R,\theta) = 0$ for all θ by choosing,

$$q'\left(\frac{s}{R}\right) = -q \quad \Rightarrow \quad \boxed{q' = -q\frac{R}{s}} \tag{2.2.18}$$

So using these values for a and q' in Eq. (2.2.13), the solution to our problem for the potential at a general point $\mathbf{r}$ outside the sphere is,

$$\phi(r,\theta) = \frac{q}{[r^2+s^2-2rs\cos\theta]^{1/2}} - \frac{qR/s}{\left[r^2+\frac{R^4}{s^2}-2r\frac{R^2}{s}\cos\theta\right]^{1/2}} \tag{2.2.19}$$

$$= \frac{q}{[r^2+s^2-2rs\cos\theta]^{1/2}} - \frac{q}{\left[\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right]^{1/2}} \tag{2.2.20}$$

As in the previous case of the flat plane, the presence of the charge q outside the sphere induces a surface charge $\sigma(\theta)$ on the surface of the sphere. Since $\mathbf{E} = 0$ inside the sphere,

$$4\pi\sigma = \mathbf{E}\cdot\hat{\mathbf{r}} = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R} \quad \Rightarrow \quad \sigma(\theta) = -\frac{q}{4\pi}\frac{1}{Rs}\frac{1-(R/s)^2}{[1+(R/s)^2-2(R/s)\cos\theta]^{3/2}} \tag{2.2.21}$$

We see that $\sigma(\theta)$ is greatest at $\theta = 0$, as one should expect, since that is the point on the surface closest to the charge q.

We can integrate to find the total charge induced on the sphere,

$$q^{\text{induced}} = 2\pi\int_0^\pi d\theta\,\sin\theta\,R^2\,\sigma(\theta) = -qR/s = q' \tag{2.2.22}$$

In general, the total induced charge on the outside surface of a grounded conductor will be the sum of all the image charges positioned inside the conductor. As you will see in a homework problem, this is not true of the induced charge on the surface of a cavity inside a grounded conductor.

The force on the charge q due to the induced surface charge σ, is just given by the electric field of the image charge q'. Since $qq' < 0$, the force is always *attractive*. We have,

$$\mathbf{F} = \frac{qq'}{(s-a)^2}\,\hat{\mathbf{z}} = \frac{-q^2(R/s)}{(s-R^2/s)^2}\,\hat{\mathbf{z}} = \frac{-q^2Rs}{(s^2-R^2)^2}\,\hat{\mathbf{z}} \tag{2.2.23}$$

Close to the surface of the sphere, we can write $s = R + d$, with $d \ll R$. Then we have

$$\mathbf{F} = \frac{-q^2 R s}{(s-R)^2(s+R)^2} = \frac{-q^2 R(R+d)}{d^2(2R+d)^2} \approx \frac{-q^2}{4d^2} \tag{2.2.24}$$

where in the last step we used $d \ll R$ to write the leading order term as d gets small. Our result above is exactly what we found for the case of a charge q a distance d in front of an infinite flat conducting plane! When q is so close to the surface that $d \ll R$, the charge does not "see" that the surface is curved, and so it looks just like a flat plane. The curvature of the spherical surface will only appear as corrections of order $(d/R)^2$.

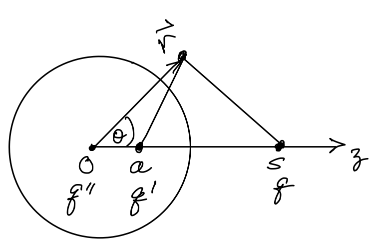
Far from the surface of the sphere, $s \gg R$, we have

$$\mathbf{F} = \frac{qq'}{(s-a)^2}\,\hat{\mathbf{z}} = \frac{-q^2 R s}{(s^2-R^2)^2}\,\hat{\mathbf{z}} \approx \frac{-q^2 R}{s^3}\,\hat{\mathbf{z}} \qquad \text{so} \qquad F \sim \frac{1}{s^3} \tag{2.2.25}$$

This is now very different from the charge in front of the infinite flat plane, as well as the force from a point charge. For both of those, the force goes as $1/(\text{distance})^2$. Here the force goes as $1/(\text{distance})^3$. The reason is that the force between q and q' goes like $1/(\text{distance})^2$, but the image charge q' decreases in magnitude as 1/distance as the charge q moves further from the sphere.

Suppose now we change this problem so that the conducting sphere is no longer grounded ($\phi = 0$) but has a net charge Q on it. What is the potential $\phi(\mathbf{r})$ in this case? What will be the potential if the sphere is neutral, with $Q = 0$?

To solve this new problem we want to add a new image charge q'' that will represent this charge Q. If we use our solution for the case of the grounded sphere to put an image charge $q' = -qR/s$ at position $a = R/s$ along the $\hat{\mathbf{z}}$ axis as before, then the $\mathbf{E}$ field from q and the image q' will have many of the properties we desire, in particular $\mathbf{E}$ will be perpendicular to the surface of the sphere at $r = R$. However the total charge on the sphere in that case will be, as we computed above, just q' and not Q. So we need our image charge to make up for the missing charge on the sphere, and so we must have $\boxed{q'' = Q - q'}$. Then the total induced charge on the sphere will be the sum of the image charges, and so be $q' + q'' = Q$, as desired. Where can we put this image charge q'' so that it does not mess up the property that $\mathbf{E}$ is perpendicular to the surface? The only place to put q'', so that its $\mathbf{E}$ field is everywhere perpendicular to the surface, is at the center of the sphere.



What real physical charge does this q'' correspond to? Just as the image q' corresponded to an induced surface charge $\sigma(\theta)$ on the surface of the sphere as given in Eq. (2.2.21), the image charge q'' corresponds to a uniformly distributed surface charge of charge density $\tilde{\sigma} = q''/4\pi R^2$ that adds to the σ from q'. The total electrostatic potential in the problem is then the sum of the electrostatic potentials from the real charge q, the surface charge σ, and the surface charge $\tilde{\sigma}$. Outside the sphere, this will be exactly equal to the point charge Coulomb potentials from the real charge q, and the image charges q' and q''. We thus have

$$\phi(r,\theta) = \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} + \frac{q'}{(r^2+a^2-2ra\cos\theta)^{1/2}} + \frac{q''}{r} \tag{2.2.26}$$

$$= \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2} + R^2 - 2rs\cos\theta\right)^{1/2}} + \frac{Q+qR/s}{r} \tag{2.2.27}$$

The force between q and the charged sphere is just the force between q and the images q'' and q'.

$$\mathbf{F} = F\hat{\mathbf{z}} = \frac{qq''}{s^2}\,\hat{\mathbf{z}} + \frac{qq'}{(s-a)^2}\,\hat{\mathbf{z}} = \frac{q(Q+qR/s)}{s^2}\,\hat{\mathbf{z}} - \frac{q^2(R/s)}{(s-(R^2/s))^2}\,\hat{\mathbf{z}} \tag{2.2.28}$$

$$F = \frac{qQ}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s-R^2/s)^2} = \frac{qQ}{s^2} + q^2R\left[\frac{1}{s^3} - \frac{1}{s^3(1-R^2/s^2)^2}\right] \tag{2.2.29}$$

$$= \frac{qQ}{s^2} + \frac{q^2R}{s^3}\left[1 - \frac{1}{(1-R^2/s^2)^2}\right] = \frac{qQ}{s^2} - \frac{q^2R^3}{s^5}\frac{(2-R^2/s^2)}{(1-R^2/s^2)^2} \tag{2.2.30}$$

Far from the sphere, $s \gg R$, the leading term is $F = qQ/s^2$, which is just what one would expect if the sphere was a point charge Q. The second term in the above gives the correction to this leading term which arises due to the fact that Q is not smeared uniformly over the surface of the sphere.

Now let's consider the special case of a *neutral* conducting sphere with $Q = 0$. The force then becomes,

$$F = -\frac{q^2R^3}{s^5}\frac{(2-R^2/s^2)}{(1-R^2/s^2)^2} \tag{2.2.31}$$

For large $s \gg R$, far from the neutral sphere, we thus have

$$F \approx -\frac{2q^2R^3}{s^5} \sim \frac{1}{s^5} \tag{2.2.32}$$

which can be compared to the force far from the *grounded* sphere, as given by Eq. (2.2.25),

$$F \approx -\frac{q^2R}{s^3} \sim \frac{1}{s^3} \tag{2.2.33}$$

We thus see that there is a very big difference between the force on a charge far from a grounded vs a neutral conducting sphere! The force from the neutral sphere decays much more rapidly.

We can give a simple explanation for the $F \sim 1/s^5$ behavior for a neutral sphere. When the total charge $Q = 0$, the image charge at the origin is just $q'' = -q'$. So the two image charges, $q'' = -q'$ at the origin, and q' at a, form an electric dipole with dipole moment $p = q'a$. As we will soon see in Notes 2-4, the electric field far from a dipole decays as $E \sim p/r^3$, hence the force on the real charge q due to the electric field of the image charge dipole is $F = qE \sim qp/s^3$. But since $q' = -qR/s$ and $a = R^2/s$, we have $p = -qR^3/s^2$. So the force on q goes as $F \sim qp/s^3 = -q^2R^3/s^5$. The additional factor of 2 in Eq. (2.2.32) comes from the exact form of the dipole field, as we will see in Notes 2-4.

Return now to the more general case of $Q \neq 0$. For $s \gg R$, far from the sphere, we have,

$$F \approx \frac{qQ}{s^2} - \frac{2q^2R^3}{s^5} \tag{2.2.34}$$

For $qQ > 0$, i.e. the charge q and the charge Q on the sphere have the same sign, the first term is positive and dominates when the charge q is far enough away from the sphere. In this case $F > 0$ and the force on the charge q is *repulsive.*

However close to the surface, where $s = R + d$, from Eq. (2.2.30) we can write,

$$F = \frac{qQ}{s^2} - \frac{q^2R^3}{s^5}\frac{s^2}{R^2}\frac{\left(\dfrac{2s^2}{R^2} - 1\right)}{\left(\dfrac{s^2}{R^2} - 1\right)^2} \tag{2.2.35}$$

To lowest order in $d/R \ll 1$, we can take $s \to R$ everywhere *except* in the denominator of the second term. To lowest order in d/R we get,

$$F \approx \frac{qQ}{R^2} - \frac{q^2}{R^2}\frac{1}{\left((1+d/R)^2 - 1\right)^2} = \frac{qQ}{R^2} - \frac{q^2}{R^2}\frac{1}{(-2d/R)^2} = \frac{qQ}{R^2} - \frac{q^2}{4d^2} \tag{2.2.36}$$

The first term is just the Coulomb repulsion between the charge q just outside the surface, and the charge Q on the sphere. The second term is exactly the same as we got from the charge q close to the grounded sphere. As $d \to 0$, the first term stays finite, while the second term diverges. Thus the second term dominates the first when q is sufficiently close to the sphere. In that case,

$$F \approx \frac{-q^2}{4d^2} \qquad \text{and the force on } q \text{ is } \textit{attractive} \tag{2.2.37}$$

Sufficiently close to the sphere there is no difference between the charged sphere, the neutral sphere ($Q = 0$), and the grounded sphere ($\phi(R, \theta) = 0$). This is because, no matter what is the magnitude of the image charge q'' (note $q'' = 0$ for the grounded sphere), the image charge q' lies so much closer to q than does q'', that the force between q and q' dominates.

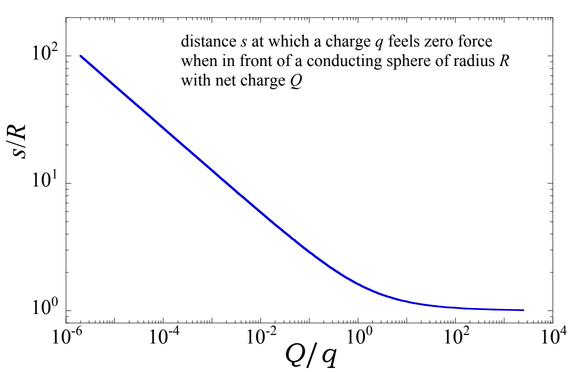
The force we computed above is one of the key components contributing to the *work function* of a metal; this is the energy one needs to supply to eject an electron from a metal. One might think that if one took a neutral piece of metal and tried to eject an electron, that there would be a significant force between the electron and the net positive charge left behind in the metal. From the above, we see that such an interaction is not the dominant effect. The dominate work we have to do is to act against the force between q and the image charge q' that is created near the surface of the metal as we try to pull off the electron. Whether the metal is charged or neutral is a much less significant effect.

For the case where q and Q have the same sign, we saw (see Eq. (2.2.34)) that the force on the charge q is repulsive if q is sufficiently far from the surface of the sphere, but the force is attractive if q is sufficiently close (see Eq. (2.2.37)). So there must be some value of s where the force goes to zero. For larger s the charge q is repelled, but for smaller s the charge q is attracted.

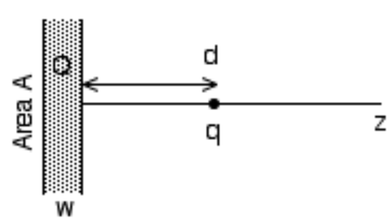
From Eq. (2.2.30), the value of s where $F = 0$ is given by,

$$F = 0 \quad \Rightarrow \quad \frac{Q}{q} = x^3 \frac{(2 - x^2)}{(1 - x^2)^2} \quad \text{where} \quad x = R/s \tag{2.2.38}$$

In the figure below we plot this distance s/R vs Q/q. When $Q/q = 1$, the crossover between an attractive and a repulsive interaction is at $s/R = 1.6$, or $s = 1.6R$. When $Q/q = 0.1$, the crossover is at $s/R = 2.8$ or $s = 2.8R$. The smaller is Q/q, the larger is the distance at which the crossover occurs.



Discussion Question 2.2.2



Consider a point charge q a distance d in front of a thin conducting plane of thickness w, as shown in the diagram. The plane has a fixed net charge Q on it. What is the electric field to the right of the plane, to the left of the plane, and inside the plane? What is the force between the charge q and the plane? Assume that the side area of the plane A is finite, so that the average surface charge $Q/2A$ is finite, however you make work the problem ignoring edge effects, i.e. assuming the plane is effectively infinite.