Unit 4-2: Electromagnetic Momentum Density and the Maxwell Stress Tensor

In this section we want to extend the notion of momentum conservation to electromagnetic fields. The math is a bit more complicated, but the idea is the same as in the last section on conservation of energy. Again, in this section, $\mathbf{B}$ and $\mathbf{E}$ are the microscopic fields and ρ and $\mathbf{j}$ are the microscopic charge and current densities.

For charges q_i at positions $\mathbf{r}_i$ with velocities $\mathbf{v}_i$, the change in the mechanical momentum $\mathbf{P}_{\text{mech}}$ of the charges due to the electromagnetic forces acting on them is,

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \sum_i \mathbf{F}_i = \sum_i q_i \left[\mathbf{E}(\mathbf{r}_i) + \frac{\mathbf{v}_i}{c} \times \mathbf{B}(\mathbf{r}_i)\right] = \int_V d^3r \left[\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right] \tag{4.2.1}$$

Using Gauss' law to write ρ in terms of $\mathbf{E}$, and Ampere's law to write $\mathbf{j}$ in terms of $\mathbf{B}$ and $\mathbf{E}$,

$$\rho = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4\pi} \qquad \text{and} \qquad \mathbf{j} = \frac{c}{4\pi}\left[\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right] \tag{4.2.2}$$

we have,

$$\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B} = \frac{1}{4\pi}\left[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \left(\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}\right] \tag{4.2.3}$$

Now

$$\frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t} = \frac{1}{c}\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right) + \frac{1}{c}\left(\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\right) = \frac{1}{c}\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) \tag{4.2.4}$$

where in the last step we used Faraday's law, $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$ to rewrite the second term.

So we have,

$$-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} = -\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t} \tag{4.2.5}$$

Substituting into Eq. (4.2.3) we then get,

$$\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B} = \frac{1}{4\pi}\left[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}\right] \tag{4.2.6}$$

where we wrote $(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = -\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})$, and added the term $\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) = 0$ in order to make the expression look symmetric with respect to interchanging $\mathbf{E}\leftrightarrow\mathbf{B}$.

We now define the *electromagnetic momentum density* vector $\boldsymbol{\Pi}$,

$$\boxed{\boldsymbol{\Pi} \equiv \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\mathbf{S} \qquad \text{with } \mathbf{S} \text{ the Poynting vector}} \tag{4.2.7}$$

Then we can write,

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} + \frac{d}{dt}\int_V d^3r\,\boldsymbol{\Pi} = \frac{1}{4\pi}\int_V d^3r\,\left[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})\right] \tag{4.2.8}$$

The goal is now to write the right hand side as the flux of a momentum current through the surface $\mathcal{S}$ bounding V.

The ith component of the electric field part of the integrand on the right hand side is,

$$E_i\partial_j E_j - \epsilon_{ijk}E_j\epsilon_{klm}\partial_l E_m \tag{4.2.9}$$

where we use the convention that repeated indices are summed over, $\partial_j \equiv \partial/\partial r_j$, and the Levi-Civita symbol ϵ_{ijk} is used for the cross products.

Using the relation $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (you should remember this!), we then get,

$$E_i\partial_j E_j - \epsilon_{ijk}E_j\epsilon_{klm}\partial_l E_m = E_i\partial_j E_j - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})E_j\partial_l E_m \tag{4.2.10}$$

$$= E_i\partial_j E_j - E_j\partial_i E_j + E_j\partial_j E_i \tag{4.2.11}$$

$$= \partial_j\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) \tag{4.2.12}$$

A similar expression holds for the magnetic field part.

We now define the *Maxwell Stress Tensor* $\overleftrightarrow{\mathbf{T}}$,

$$\overleftrightarrow{\mathbf{T}} = \frac{1}{4\pi}\left[\mathbf{EE} + \mathbf{BB} - \frac{1}{2}\left(E^2 + B^2\right)\mathbf{I}\right] \tag{4.2.13}$$

or in terms of components,

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\left(E^2 + B^2\right)\delta_{ij}\right]} \tag{4.2.14}$$

Note, $\overleftrightarrow{\mathbf{T}}$ is a symmetric tensor, with $T_{ij} = T_{ji}$.

So now we have for the ith component of momentum,

$$\frac{dP_{\text{mech,i}}}{dt} + \frac{d}{dt}\int_V d^3r\,\Pi_i = \int_V d^3r\,\partial_j T_{ij} = \oint_{\mathcal{S}} da\,T_{ij}\cdot\hat{\mathbf{n}}_j \tag{4.2.15}$$

where $\partial_j T_{ij} \equiv \dfrac{\partial T_{ij}}{\partial r_j}$. We can also write the above in vector form,

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} + \frac{d}{dt}\int_V d^3r\,\mathbf{\Pi} = \oint_{\mathcal{S}} da\,\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} \tag{4.2.16}$$

Because of the historical definition of $\overleftrightarrow{\mathbf{T}}$ as above, there is no minus sign on the right hand side of the above equation, which leads to the interpretation that $-\overleftrightarrow{\mathbf{T}}$ is the current of electromagnetic momentum. In particular, $-T_{ij}$ gives the flux of the ith component of electromagnetic momentum through a unit element of surface area with normal in the $\hat{\mathbf{e}}_j$ direction.

Note, $\dfrac{d\mathbf{P}_{\text{mech}}}{dt} = \sum_i \mathbf{F}_i \equiv \mathbf{F}_{\text{EM}}$ is the total electromagnetic force on the volume V. Hence we can write,

$$\mathbf{F}_{\text{EM}} = \oint_{\mathcal{S}} da\,\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} - \frac{d}{dt}\int_V d^3r\,\mathbf{\Pi} \tag{4.2.17}$$

For a *static* situation, $\mathbf{\Pi}$ is constant in time and the second term above vanishes. We then have,

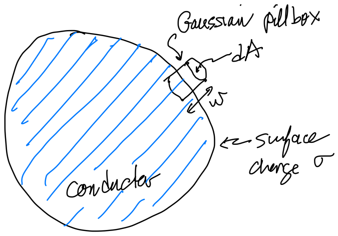
$$\mathbf{F}_{\text{EM}} = \oint_{\mathcal{S}} da\,\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} \qquad \text{for statics} \tag{4.2.18}$$

so from this we conclude that T_{ij} is the ith component of the static force on a unit surface area with normal $\hat{\mathbf{e}}_j$, where $\hat{\mathbf{e}}_j$ is pointing *outwards* from the volume the force $\mathbf{F}_{\text{EM}}$ is acting upon.

This is the origin of the term "stress" tensor. $\overleftrightarrow{\mathbf{T}}$ is like the stress tensor of an elastic medium, where T_{xx}, T_{yy}, and T_{zz} are like pressure, and the off diagonal elements are like shear stresses.

Force on a Conductor Surface

Using the result above, we can now compute the net electrostatic force acting on a conductor's surface. Recall, for a conductor in an electrostatic situation, all the charge lies on the conductor's surface. Consider a conductor as sketched below.



Consider a small Gaussian pillbox piercing the surface, as shown in the sketch. The pillbox has top and bottom areas dA and width w. By Eq. (4.2.18), the electromagnetic force on the volume contained within the pillbox is

$$\mathbf{F}_{\rm EM} = \oint_{\mathcal{S}} da\, \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}} \tag{4.2.19}$$

where $\mathcal{S}$ is the surface of the pillbox. As we shrink the width $w \to 0$, this will give the net force on the patch of surface of area dA with surface charge density σ.

As the side width $w \to 0$, the contribution to the surface integral in Eq. (4.2.19) from this side will vanish. Since we are in an electrostatic situation, we only need to consider the part of $\overleftrightarrow{\mathbf{T}}$ that depends on the electric field $\mathbf{E}$. Inside the conductor $\mathbf{E} = 0$, and so the contribution to the integral in Eq (4.2.19) from the bottom side of the pillbox will also vanish. All that remains is the contribution from the top side. We thus get $\mathbf{F}_{\rm EM} = dA\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}$, where $\overleftrightarrow{\mathbf{T}}$ is evaluated just above the surface of the conductor. The force per unit area on the charged surface of the conductor is thus, $\mathbf{f}_{\rm EM} = \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward pointing normal to the surface of the conductor.

Using the expression for $\overleftrightarrow{\mathbf{T}}$ in Eq. (4.2.14), the force per unit surface area is,

$$\mathbf{f}_{\rm EM} = \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}} = \frac{1}{4\pi}\left[\mathbf{E}(\mathbf{E}\cdot\hat{\mathbf{n}}) - \frac{1}{2}E^2\hat{\mathbf{n}}\right] \tag{4.2.20}$$

For a conductor surface, we know that $\mathbf{E}^{\rm above} \cdot \hat{\mathbf{n}} = 4\pi\sigma$ (since $\mathbf{E}^{\rm below} = 0$). And since the tangential component of $\mathbf{E}$ at the surface of the conductor must be zero, the field just above the surface is,

$$\mathbf{E} = 4\pi\sigma\,\hat{\mathbf{n}} \tag{4.2.21}$$

Using this in the expression for the force per unit area then gives,

$$\mathbf{f}_{\rm EM} = \frac{1}{4\pi}\left[(4\pi\sigma\,\hat{\mathbf{n}})(4\pi\sigma) - \frac{1}{2}(4\pi\sigma)^2\,\hat{\mathbf{n}}\right] = 2\pi\sigma^2\,\hat{\mathbf{n}} = \frac{1}{2}\sigma\,\mathbf{E} \tag{4.2.22}$$

Note in particular the factor 1/2 in the last result. One might naively have thought that it should be $\mathbf{f}_{\rm EM} = \sigma\,\mathbf{E}$, just like the force on a point charge is $q\mathbf{E}$. But one needs to exclude the self field of the charge on the surface from acting on itself! If one considered the force on a small patch of the surface of area dA as arising from all the charge on the surface *except* for the charge on the patch dA, one arrives as the same result as above. See Griffiths Sec. 2.5.3. Alternatively, we could say that the electric field that exerts the force on the surface is the average of the field above with the field below, $\mathbf{E}_{\rm ave} = \frac{1}{2}(\mathbf{E}^{\rm above} - \mathbf{E}^{\rm below})$.

Note, the force on the conductor surface is always pointing *outwards*. If we considered a simple model of an electron to be a spherical shell of radius R, with surface charge $\sigma = -e/(4\pi R^2)$, then the electron would in a sense be unstable – there would be the force $2\pi\sigma^2\,\hat{\mathbf{n}}$ pushing the surface outwards. Since the electron is stable, there must be some other mechanical forces holding the electron together, to balance out this electromagnetic force from the Maxwell stress tensor. These mysterious "other" forces are called the Poincaré stresses.