Unit 6-3: Radiation in the Electric Dipole Approximation - Beyond the Radiation Zone

From Eq. (6.3.8) we can write,

$$\mathbf{E}_{E1} = -ik\,\boldsymbol{\nabla}\times\left[\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\mathbf{p}_\omega\times\hat{\mathbf{r}}\right] \qquad (6.3.S.1)$$

This is the correct expression *before* we have taken the radiation zone approximation.

We now wish to work out all the above derivatives so as to arrive at Eq. (6.3.12).

To evaluate $\boldsymbol{\nabla}\times[\cdots]$ we use $\boldsymbol{\nabla}\times[f\mathbf{g}] = f\boldsymbol{\nabla}\times\mathbf{g} + [\boldsymbol{\nabla}f]\times\mathbf{g}$, with $f = \dfrac{e^{ikr}}{r}\left(1+\dfrac{i}{kr}\right)$ and $\mathbf{g} = \mathbf{p}_\omega\times\hat{\mathbf{r}}$.

$$\boldsymbol{\nabla}\times[\cdots] = \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\boldsymbol{\nabla}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}}) + \boldsymbol{\nabla}\left[\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right]\times(\mathbf{p}_\omega\times\hat{\mathbf{r}}) \qquad (6.3.S.2)$$

Let's evaluate the first factor of the second term. We can take the gradient using spherical coordinates, noting that f depends only on the radial coordinate r. We have,

$$\boldsymbol{\nabla}\left[\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right] = \frac{d}{dr}\left[\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right]\hat{\mathbf{r}} = e^{ikr}\left[ik\left(\frac{1}{r}+\frac{i}{kr^2}\right) - \frac{1}{r^2} - \frac{2i}{kr^3}\right]\hat{\mathbf{r}} \qquad (6.3.S.3)$$

$$= \frac{e^{ikr}}{r}\left[ik - \frac{2}{r} - \frac{2i}{kr^2}\right]\hat{\mathbf{r}} \qquad (6.3.S.4)$$

Now let's evaluate the last factor of the first term,

$$\boldsymbol{\nabla}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}}) = \mathbf{p}_\omega(\boldsymbol{\nabla}\cdot\hat{\mathbf{r}}) - (\mathbf{p}_\omega\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} \qquad (6.3.S.5)$$

Now, evaluating in spherical coordinates, we have,

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{r}} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\right) = \frac{2}{r} \qquad (6.3.S.6)$$

and

$$(\mathbf{p}_\omega\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} = \sum_{k=1}^{3} p_{\omega\,k}\,\frac{\partial\hat{\mathbf{r}}}{\partial r_k} \qquad (6.3.S.7)$$

where

$$\frac{\partial\hat{\mathbf{r}}}{\partial r_k} = \frac{\partial}{\partial r_k}\left(\frac{\mathbf{r}}{r}\right) = \mathbf{r}\left(-\frac{1}{r^2}\frac{\partial r}{\partial r_k}\right) + \frac{\hat{\mathbf{e}}_k}{r} = \mathbf{r}\left(-\frac{1}{r^2}\frac{r_k}{r}\right) + \frac{\hat{\mathbf{e}}_k}{r} \qquad \text{since} \quad \frac{\partial r}{\partial r_k} = \frac{r_k}{r} \qquad (6.3.S.8)$$

Here $\hat{\mathbf{e}}_k = \dfrac{\partial\mathbf{r}}{\partial r_k}$ is the unit vector in direction k.

So putting the above pieces together we get,

$$\boldsymbol{\nabla}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}}) = \frac{2\mathbf{p}_\omega}{r} - \sum_{k=1}^{3} p_{\omega\,k}\left(-\frac{r_k\,\mathbf{r}}{r^3} + \frac{\hat{\mathbf{e}}_k}{r}\right) = \frac{2\mathbf{p}_\omega}{r} + \frac{(\mathbf{p}_\omega\cdot\mathbf{r})\,\mathbf{r}}{r^3} - \frac{\mathbf{p}_\omega}{r} \qquad (6.3.S.9)$$

$$= \frac{\mathbf{p}_\omega + (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r} \qquad \text{using} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \qquad (6.3.S.10)$$

Now, putting all the pieces together, we get,

$$\mathbf{E}_{E1} = -ik\,\frac{\mathrm{e}^{ikr}}{r}\left[\left(1+\frac{i}{kr}\right)\frac{\mathbf{p}_\omega + (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r} + \left(ik - \frac{2}{r} - \frac{2i}{kr^2}\right)\hat{\mathbf{r}}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}})\right] \tag{6.3.S.11}$$

The very last factor we can rewrite as, $\hat{\mathbf{r}}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}}) = \mathbf{p}_\omega - (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}$.

Rewriting the above, ordering the different terms by powers of $\dfrac{1}{r}$, we get

$$\mathbf{E}_{E1} = -ik\,\frac{\mathrm{e}^{ikr}}{r}\left[ik(\mathbf{p}_\omega - (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}) + \frac{1}{r}\left(1+\frac{i}{kr}\right)(\mathbf{p}_\omega + (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}) - \frac{2}{r}\left(1+\frac{i}{kr}\right)(\mathbf{p}_\omega - (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}})\right] \tag{6.3.S.12}$$

$$= k^2\,\frac{\mathrm{e}^{ikr}}{r}\left[\mathbf{p}_\omega - (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \frac{i}{kr}\left(1+\frac{i}{kr}\right)(\mathbf{p}_\omega + (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - 2\mathbf{p}_\omega + 2(\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}})\right] \tag{6.3.S.13}$$

And so finally,

$$\boxed{\mathbf{E}_{E1} = k^2\,\frac{\mathrm{e}^{ikr}}{r}\left[\mathbf{p}_\omega - (\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \frac{i}{kr}\left(1+\frac{i}{kr}\right)(3(\mathbf{p}_\omega\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_\omega)\right]} \tag{6.3.S.14}$$

which is just Eq. (6.3.12).