Unit 6-5: Radiation from Arbitrarily Time Varying Sources and Larmor's Formula

In the previous sections we considered the radiation from a pure harmonically oscillating source, i.e. oscillating at a single frequency ω. Here we consider a source with a general time dependence. We consider only the electric dipole approximation to the radiation, since in the long wavelength (non-relativistic) limit that is the leading term.

For $\mathbf{p}(t) = \mathbf{p}_\omega \mathrm{e}^{-i\omega t}$, a pure harmonic oscillation, we found that the radiated fields, in the electric dipole approximation, oscillate at the same frequency, $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$ and $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$, with amplitudes given by,

$$\mathbf{E}_\omega = -k^2 \frac{\mathrm{e}^{ikr}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) = -\frac{\omega^2}{c^2} \frac{\mathrm{e}^{i\omega r/c}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) \tag{6.5.1}$$

$$\mathbf{B}_\omega = k^2 \frac{\mathrm{e}^{ikr}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega = \frac{\omega^2}{c^2} \frac{\mathrm{e}^{i\omega r/c}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega \qquad \text{since } k = \omega/c \tag{6.5.2}$$

For an arbitrarily time varying charge distribution, with electric dipole moment,

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{p}_\omega \, \mathrm{e}^{-i\omega t} \tag{6.5.3}$$

the solution for the fields is obtained by superposition,

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{E}_\omega \, \mathrm{e}^{-i\omega t} = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) \tag{6.5.4}$$

$$= \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \omega^2 \mathbf{p}_\omega\right] \tag{6.5.5}$$

$$= \frac{1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \mathbf{p}_\omega\right] \tag{6.5.6}$$

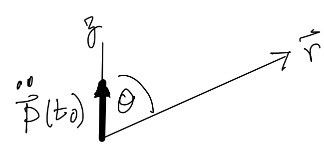
$$= \frac{1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \frac{d^2}{dt^2} \mathbf{p}(t - r/c)\right] \tag{6.5.7}$$

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{1}{c^2 r} \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t - r/c)]} \qquad \text{where} \quad \ddot{\mathbf{p}} = \frac{d^2\mathbf{p}}{dt^2} \tag{6.5.8}$$

define the *retarded time* $t_0 = t - r/c$.

In spherical coordinates, with $\ddot{\mathbf{p}}(t_0)$ aligned along $\hat{\mathbf{z}}$,

$$\mathbf{E}(\mathbf{r}, t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin\theta \, \hat{\boldsymbol{\theta}} \tag{6.5.9}$$



Similarly,

$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{B}_\omega \, \mathrm{e}^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{\mathbf{r}} \times \mathbf{p}_\omega \tag{6.5.10}$$

$$= \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \mathbf{p}_\omega \tag{6.5.11}$$

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)} \tag{6.5.12}$$

In the above spherical coordinates,

$$\mathbf{B}(\mathbf{r},t) = \frac{\ddot{p}(t_0)}{c^2\, r}\,\sin\theta\,\hat{\boldsymbol{\varphi}} \tag{6.5.13}$$

The Poynting vector is then,

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\left(\frac{1}{c^2\, r}\right)^2 [\ddot{p}(t_0)]^2\,\sin^2\theta\,\hat{\mathbf{r}} \tag{6.5.14}$$

The total power radiated through the surface $\mathcal{S}$ of a sphere of radius r is then,

$$P = \oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{S} = 2\pi\int_0^\pi d\theta\sin\theta\, r^2\,\hat{\mathbf{r}}\cdot\mathbf{S} = \frac{[\ddot{p}(t_0)]^2}{2c^3}\int_0^\pi d\theta\sin^3\theta \tag{6.5.15}$$

Using $\int_0^\pi d\theta\sin^3\theta = 4/3$ we then get,

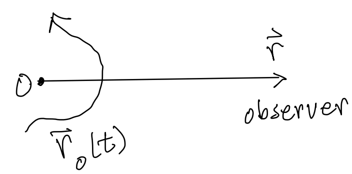
$$\boxed{P = \frac{2[\ddot{p}(t_0)]^2}{3c^3}} \tag{6.5.16}$$

Power radiated by an accelerating charge

Apply the above to a point charge q moving along the trajectory $\mathbf{r}_0(t)$. Then,

$$\mathbf{p}(t) = q\,\mathbf{r}_0(t) \qquad \text{so} \qquad \ddot{\mathbf{p}}(t) = q\,\ddot{\mathbf{r}}_0(t) = q\,\mathbf{a}(t)$$

where $\mathbf{a}$ is the charge's acceleration.



The radiated power is then,

$$\boxed{P = \frac{2}{3}\frac{q^2\,a^2(t_0)}{c^3}} \tag{6.5.17}$$

This is <u>*Larmor's formula*</u> for the power radiated by an accelerating charge.

The total power passing through a sphere of radius r at time t is due to the acceleration of the charge at the retarded time $t_0 = t - r/c$.

Note, there is only power radiated *if* the charge is accelerating! As we saw in Notes 5-1, a charge moving at constant velocity does *not* radiate.

power radiated $\sim$ (acceleration)^2

Since we derived this result using the electric dipole approximation in the long wavelength limit $\lambda \gg d$, the above Larmor's formula holds only in the limit of a *non-relativistically* moving charge, $v \ll c$. In unit 7 we will see how to extend Larmor's formula to the case where the charge may be moving relativistically fast.

Discussion Question 6.5

We derived the above expression for Larmor's formula by using the electric dipole approximation. The electric dipole approximation was the leading term in the long wavelength approximation, which can be viewed as giving an expansion in powers of $kq \sim v/c$.

If we want the analog of Larmor's formula, but now for a charge moving relativistically fast, so that v/c is not small, it suggests we would have to keep lots of higher order terms in this long wavelength approximation – we would have to keep the magnetic dipole term, the electric quadrupole term, the magnetic quadrupole term, the electric octupole term, and indeed all higher terms. Clearly we can't do that! So is there a clever way we could get the relativistic Larmor's formula without all that work?

Instability of the Classical Model of an Atom

In the classical model of an atom the electron orbits the nucleus just like a planet orbits the sun. Such an orbiting electron has centripetal acceleration, and once it was realized that accelerated charges radiate electromagnetic waves it was realized that a classical orbiting electron would radiate away its energy and spiral into the nucleus. Thus the classical model of the atom is unstable!

It is of interest to compute what is the time for such an electron to crash into the nucleus. If it were a time of order the age of the universe, maybe we would not worry. But it turns out to be a very short time!

We will assume for simplicity that the nucleus has a charge $+e$.

The motion of the electron around the nucleus is given by Newton's equation.

$$\mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \frac{e^2}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad \frac{1}{2}mv^2 = \frac{e^2}{2r} \tag{6.5.18}$$

where we used $a = v^2/r$ as the centripetal acceleration.

The total energy of the electron orbiting at radius r is the sum of its kinetic plus electrostatic potential energy,

$$E(r) = \frac{1}{2}mv^2 + V(r) = \frac{e^2}{2r} - \frac{e^2}{r} = -\frac{e^2}{2r} \tag{6.5.19}$$

Now consider the energy lost by the electron due to radiation. From Larmor's formula we have,

$$\frac{dE}{dt} = -P = -\frac{2}{3}\frac{e^2a^2}{c^3} \tag{6.5.20}$$

Using $a = v^2/r = e^2/mr^2$ then gives,

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^2}{c^3}\left(\frac{e^2}{mr^2}\right)^2 \tag{6.5.21}$$

Now,

$$\frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} = \left[\frac{d}{dr}\left(\frac{-e^2}{2r}\right)\right]\frac{dr}{dt} = \frac{e^2}{2r^2}\frac{dr}{dt} \tag{6.5.22}$$

Equating Eqs. (6.5.20) and (6.5.22) then gives,

$$\frac{e^2}{2r^2}\frac{dr}{dt} = -\frac{2}{3}\frac{e^2}{c^3}\left(\frac{e^2}{mr^2}\right)^2 \qquad \Rightarrow \qquad \frac{dr}{dt} = -\frac{4}{3c^3}\left(\frac{e^2}{m}\right)^2\frac{1}{r^2} \tag{6.5.23}$$

It is useful now to introduce the time scale

$$\tau \equiv \frac{2}{3}\frac{e^2}{mc^3} \tag{6.5.24}$$

(we can see this has units of time since e^2/r and mc^2 both have units of energy). With this definition, the above becomes

$$\frac{dr}{dt} = -\frac{3c^3\tau^2}{r^2} \tag{6.5.25}$$

Next, we integrate the above to get $r(t)$

$$\int_{r_0}^{r(t)} dr\, r^2 = -\int_0^t dt\, 3c^3\tau^2 \qquad \Rightarrow \qquad \frac{r^3(t) - r_0^3}{3} = -3c^3\tau^2 t \tag{6.5.26}$$

so

$$r^3(t) = r_0^3 - 9c^3\tau^2 t \tag{6.5.27}$$

The cube of the radius decreases linearly in time!

We can now expect that the electron will crash into the nucleus when $r(t) \approx 0$, or when,

$$t = \frac{r_0^3}{9c^3\tau^2} = \left(\frac{r_0}{c\tau}\right)^3 \frac{\tau}{9} \tag{6.5.28}$$

To see how long this is, we note that r_0 is just the Bohr radius,

$$r_0 = 0.5 \times 10^{-8}\mathrm{cm} \tag{6.5.29}$$

while

$$\tau = \frac{2}{3}\frac{e^2}{mc^3} = 6.26 \times 10^{-24}\mathrm{sec} \tag{6.5.30}$$

and

$$c\tau = 2 \times 10^{-13}\mathrm{cm} \qquad \text{this is the typical length scale of a heavy atomic nucleus} \tag{6.5.31}$$

Putting it together we get,

$$t = \left(\frac{0.5 \times 10^{-8}}{2 \times 10^{-13}}\right)^3 \frac{6 \times 10^{-24}}{9}\mathrm{sec} \approx 10^{-11}\mathrm{sec} \tag{6.5.32}$$

So the atom would go unstable on an extremely short time scale!

This was a fundamental problem that faced physics at the end of the 1800s. Given that charges radiate when they are accelerated, the classical model of the electron orbiting the nucleus cannot be stable!

The solution to this paradox was provided by quantum mechanics. In quantum mechanics you learn that the stable quantized energy states of the atom are the solutions to the *time-independent* Schrödinger equation. The solutions are *time-independent* wavefunctions $\psi(\mathbf{r})$, and the charge density of such an electron, $\rho(\mathbf{r}) = -e|\psi(\mathbf{r})|^2$, is therefore independent of time. The electrons in such stable eigenstates are *not* being accelerated, so they do not radiate electromagnetic energy!