Unit 7-5: The Relativistic Maxwell Stress Tensor

Consider the 2nd rank symmetric 4-tensor, constructed from the field strength tensor $F_{\mu\nu}$ by,

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F_{\lambda\nu} - \frac{1}{4}\delta_{\mu\nu}F_{\lambda\sigma}F_{\sigma\lambda}\right] \tag{7.5.1}$$

Let us evaluate the pieces,

$$F_{\mu\lambda}F_{\lambda\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{7.5.2}$$

$$= \begin{pmatrix} -B_3^2 - B_2^2 + E_1^2 & B_1B_2 + E_1E_2 & B_1B_3 + E_1E_3 & -iE_2B_3 + iE_3B_2 \\ B_1B_2 + E_1E_2 & -B_3^2 - B_1^2 + E_2^2 & B_3B_2 + E_3E_2 & -iE_3B_1 + iE_1B_3 \\ B_1B_3 + E_1E_3 & B_3B_2 + E_3E_2 & -B_2^2 - B_1^2 + E_3^2 & -iE_1B_2 + iE_2B_1 \\ -iE_2B_3 + iE_3B_2 & -iE_3B_1 + iE_1B_3 & -iE_1B_2 + iE_2B_1 & E_1^2 + E_2^2 + E_3^2 \end{pmatrix} \tag{7.5.3}$$

And from the trace of the above matrix we get

$$\frac{1}{4}F_{\lambda\sigma}F_{\sigma\lambda} = \frac{1}{2}\left[E^2 - B^2\right] \tag{7.5.4}$$

Putting the pieces together we get

$$T_{\mu\nu} = \frac{1}{4\pi}\begin{pmatrix} E_1^2 + B_1^2 - \frac{1}{2}[E^2+B^2] & E_1E_2 + B_1B_3 & E_1E_3 + B_1B_3 & -i(\mathbf{E}\times\mathbf{B})_1 \\ E_1E_2 + B_1B_2 & E_2^2 + B_2^2 - \frac{1}{2}[E^2+B^2] & E_2E_3 + B_2B_3 & -i(\mathbf{E}\times\mathbf{B})_2 \\ E_1E_3 + B_1B_3 & E_2E_3 + B_2B_3 & E_3^2 + B_3^2 - \frac{1}{2}[E^2+B^2] & -i(\mathbf{E}\times\mathbf{B})_3 \\ -i(\mathbf{E}\times\mathbf{B})_1 & -i(\mathbf{E}\times\mathbf{B})_2 & -i(\mathbf{E}\times\mathbf{B})_3 & \frac{1}{2}[E^2+B^2] \end{pmatrix} \tag{7.5.5}$$

From which we can identify the components as,

$$T_{\mu\nu} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & -ic\Pi_1 \\ T_{21} & T_{22} & T_{23} & -ic\Pi_2 \\ T_{23} & T_{23} & T_{33} & -icp\Pi_3 \\ -ic\Pi_1 & -ic\Pi_2 & -ic\Pi_3 & u \end{pmatrix} \tag{7.5.6}$$

where u is the electromagnetic energy density, $\mathbf{\Pi}$ is the electromagnetic momentum density, and T_{ij} is the Maxwell stress tensor. Thus we see that these are all different parts of a single symmetric 2nd rank 4-tensor. Note, from Eq. (7.5.5) we see that the trace of $T_{\mu\nu}$ vanishes, i.e. $T_{\mu\mu} = 0$.

Since $T_{\mu\nu}$ is a 4-tensor, we can get a 4-vector by taking its inner product with the 4-gradient, $\dfrac{\partial T_{\mu\nu}}{\partial x_\nu}$. Consider the temporal component of this 4-vector,

$$\frac{\partial T_{4\nu}}{\partial x_\nu} = \left(\mathbf{\nabla},\ \frac{\partial}{ic\partial t}\right)\cdot(-ic\mathbf{\Pi},\ u) = -ic\mathbf{\nabla}\cdot\mathbf{\Pi} + \frac{\partial u}{ic\partial t} = -\frac{i}{c}\left(c^2\mathbf{\nabla}\cdot\mathbf{\Pi} + \frac{\partial u}{\partial t}\right) = -\frac{i}{c}\left(\mathbf{\nabla}\cdot\mathbf{S} + \frac{\partial u}{\partial t}\right) \tag{7.5.7}$$

where $\mathbf{S} = c^2\mathbf{\Pi}$ is the Poynting vector. From our discussion of energy conservation in Notes 4-1 we know that,

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{7.5.8}$$

where u_{mech} is the density of mechanical energy (i.e. kinetic energy) of the charges. Thus we have,

$$\frac{\partial T_{4\nu}}{\partial x_\nu} = \frac{i}{c}\frac{\partial u_{\text{mech}}}{\partial t} \tag{7.5.9}$$

Consider now the spatial components,

$$\frac{\partial T_{i\nu}}{\partial x_\nu} = \left(\nabla,\ \frac{\partial}{ic\partial t}\right) \cdot (T_{ij},\ -ic\Pi_i) = [\nabla \cdot \overleftrightarrow{\mathbf{T}}]_i - \frac{\partial \Pi_i}{\partial t} \tag{7.5.10}$$

From our discussion of momentum conservation in Notes 4-2 we know that,

$$\nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial \mathbf{\Pi}}{\partial t} = \frac{\partial \mathbf{\Pi}_{\text{mech}}}{\partial t} \tag{7.5.11}$$

where $\mathbf{\Pi}_{\text{mech}}$ is the density of mechanical momentum of the charges.

Consider the right hand sides of Eqs. (7.5.9) and (7.5.11). From Notes 4-1 we know that,

$$\frac{i}{c}\frac{\partial u_{\text{mech}}}{\partial t} = \frac{i}{c}\,\mathbf{E} \cdot \mathbf{j} \tag{7.5.12}$$

And from Notes 4-2 we know that,

$$\frac{\partial \mathbf{\Pi}_{\text{mech}}}{\partial t} = \rho\,\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B} \tag{7.5.13}$$

These terms are just the temporal and spatial components of $\frac{1}{c}F_{\mu\nu}j_\nu$, as we see as follows. With,

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \qquad \text{and} \qquad j_\mu = (\mathbf{j},\ ic\rho) \tag{7.5.14}$$

we have for the temporal component,

$$\frac{1}{c}F_{4\nu}j_\nu = \frac{i}{c}\,\mathbf{E} \cdot \mathbf{j} \tag{7.5.15}$$

And for the spatial x-component we have,

$$\frac{1}{c}F_{1\nu}j_\nu = \frac{1}{c}\left(B_3 j_2 - B_2 j_3 + c\rho\,E_1\right) = \left(\frac{\mathbf{j}}{c} \times \mathbf{B}\right)_1 + \rho\,E_1 \tag{7.5.16}$$

We thus conclude that energy and momentum conservation can be written in a relativistic form as,

$$\boxed{\frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{1}{c}F_{\mu\nu}j_\nu} \tag{7.5.17}$$

Since the above is an equality between two 4-vectors, we know that each side of this equation transforms the same way under a Lorentz transformation. We can thus conclude that if energy and momentum are conserved in one inertial frame of reference, they must be conserved in all inertial frames of reference.

As an alternative way to derive the above, consider the right hand sides of Eqs. (7.5.9) and (7.5.11). Since these are the temporal and spatial parts of $\partial T_{\mu\nu}/\partial x_\nu$, we know they constitute a 4-vector,

$$\frac{\partial}{\partial t}\left(\mathbf{\Pi}_{\rm mec}, \frac{iu_{\rm mec}}{c}\right) = \frac{1}{\gamma}\frac{\partial}{\partial s}\left(\mathbf{\Pi}_{\rm mec}, \frac{iu_{\rm mec}}{c}\right) \qquad \text{where } ds = dt/\gamma \text{ is the proper time interval} \tag{7.5.18}$$

This looks almost like $\partial/\partial s$ of the energy-momentum 4-vector except for two things: the prefactor of $1/\gamma$, and the fact that $\mathbf{\Pi}_{\rm mech}$ and $u_{\rm mec}$ are *densitites.* To see the meaning of this 4-vector, consider the total momentum $\mathbf{p}_{\rm mech}$ and total energy $\mathcal{E}_{\rm mech}$ of the charges contained in a small box of volume ΔV. This does give a 4-vector,

$$\frac{d}{ds}\left(\mathbf{p}_{\rm mech}, \frac{i\mathcal{E}_{\rm mech}}{c}\right) = \Delta V \frac{\partial}{\partial s}\left(\mathbf{\Pi}_{\rm mech}, \frac{iu_{\rm mech}}{c}\right) \tag{7.5.19}$$

So from this we see that,

$$\frac{1}{\gamma}\frac{\partial}{\partial s}\left(\mathbf{\Pi}_{\rm mec}, \frac{iu_{\rm mec}}{c}\right) = \frac{1}{\gamma \Delta V}\frac{d}{ds}\left(\mathbf{p}_{\rm mech}, \frac{i\mathcal{E}_{\rm mech}}{c}\right) = \frac{1}{\Delta \mathring{V}}\frac{d}{ds}\left(\mathbf{p}_{\rm mech}, \frac{i\mathcal{E}_{\rm mech}}{c}\right) \tag{7.5.20}$$

where $\Delta\mathring{V}$ is the volume of the box in the rest frame of the charge it contains (see the discussion of the 4-current in Notes 7-2).

Now, $dp_\mu/ds = K_\mu$ is the 4-force, which in this case is the 4-Lorentz force $K_\mu = (\Delta Q/c)F_{\mu\nu}u_\nu$, where ΔQ is the charge in the box ΔV, and u_ν is its 4-velocity. Then we have,

$$\frac{1}{\Delta\mathring{V}}\frac{d}{ds}\left(\mathbf{p}_{\rm mech}, \frac{i\mathcal{E}_{\rm mech}}{c}\right) = \frac{1}{\mathring{V}}K_\mu = \frac{1}{c}\frac{\Delta Q}{\Delta \mathring{V}}\, F_{\mu\nu}\, u_\nu = \frac{1}{c}\, F_{\mu\nu}\, \mathring{\rho}\, u_\nu = \frac{1}{c}\, F_{\mu\nu}\, j_\nu \tag{7.5.21}$$

where from Notes 7-3 we used $\mathring{\rho} = \Delta Q/\Delta\mathring{V}$, and $j_\nu = \mathring{\rho} u_\nu$.

Thus we conclude,

$$\frac{\partial}{\partial t}\left(\mathbf{\Pi}_{\rm mech}, \frac{iu_{\rm mech}}{c}\right) = \frac{1}{c}\, F_{\mu\nu}\, j_\nu \tag{7.5.22}$$