Unit 7-6: Poincaré Stresses and the 4/3 Problem

In Unit 7-4 we derived the relativistic Larmor's formula by assuming that, for an accelerating point charge, the total electromagnetic momentum radiated $\boldsymbol{\mathcal{P}}$ and the total electromagnetic energy radiated $\mathcal{E}$ were the space and time components of an energy-momentum 4-vector, $\mathcal{P}_\mu = (\boldsymbol{\mathcal{P}}, i\mathcal{E}/c)$.

But in Unit 7-5 we saw that the electromagnetic energy density u and the electromagnetic momentum density $\mathbf{\Pi}$ were in fact the time components of the relativistic Maxwell Stress Tensor $T_{\mu\nu}$. As such, the total energy $\mathcal{E} = \int d^3r\, u(\mathbf{r})$ and the total momentum $\boldsymbol{\mathcal{P}} = \int d^3r\, \mathbf{\Pi}(\mathbf{r})$ should be governed by the Lorentz transformation properties of the 4-tensor $T_{\mu\nu}$, which are *not* necessarily the same as a 4-vector!

Below we investigate this situation and show how to resolve this seeming conflict using the notion of Poincaré stresses.

Lorentz transformation of the relativistic Maxwell Stress Tensor

If frame $\mathcal{K}'$ moves with $\mathbf{v} = v\hat{\mathbf{x}}$ with respect to frame $\mathcal{K}$, then the Lorentz transformation from $\mathcal{K}$ to $\mathcal{K}'$ is,

$$a_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{7.6.1}$$

The energy-stress 4-tensor transforms like,

$$T'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} T_{\sigma\lambda} = a_{\mu\sigma} T_{\sigma\lambda} a^t_{\lambda\nu} \tag{7.6.2}$$

$$T'_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} & T_{x4} \\ T_{yx} & T_{yy} & T_{yz} & T_{y4} \\ T_{zx} & T_{zy} & T_{zz} & T_{z4} \\ T_{4x} & T_{4y} & T_{4z} & T_{44} \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{7.6.3}$$

$$T'_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \left(\begin{array}{c|c|c|c} \gamma T_{xx} + i\frac{v}{c}\gamma T_{x4} & T_{xy} & T_{xz} & -i\frac{v}{c}\gamma T_{xx} + \gamma T_{x4} \\ \gamma T_{yx} + i\frac{v}{c}\gamma T_{y4} & T_{yy} & T_{yz} & -i\frac{v}{c}\gamma T_{yx} + \gamma T_{y4} \\ \gamma T_{zx} + i\frac{v}{c}\gamma T_{z4} & T_{zy} & T_{zz} & -i\frac{v}{c}\gamma T_{zx} + \gamma T_{z4} \\ \gamma T_{4x} + i\frac{v}{c}\gamma T_{44} & T_{4y} & T_{4z} & -i\frac{v}{c}\gamma T_{4x} + \gamma T_{44} \end{array}\right) \tag{7.6.4}$$

$$T'_{\mu\nu} = \left(\begin{array}{c|c|c|c} \gamma^2 T_{xx} + i\frac{v}{c}\gamma^2 T_{x4} + i\frac{v}{c}(\gamma^2 T_{4x} + i\frac{v}{c}\gamma^2 T_{44}) & \gamma T_{xy} + i\frac{v}{c}\gamma T_{4y} & \gamma T_{xz} + i\frac{v}{c}\gamma T_{4z} & -i\frac{v}{c}\gamma^2 T_{xx} + \gamma^2 T_{x4} + i\frac{v}{c}(-i\frac{v}{c}\gamma^2 T_{4x} + \gamma^2 T_{44}) \\ \hline \gamma T_{yx} + i\frac{v}{c}\gamma T_{y4} & T_{yy} & T_{yz} & -i\frac{v}{c}\gamma T_{yx} + \gamma T_{y4} \\ \hline \gamma T_{zx} + i\frac{v}{c}\gamma T_{z4} & T_{zy} & T_{zz} & -i\frac{v}{c}\gamma T_{zx} + \gamma T_{z4} \\ \hline -i\frac{v}{c}(\gamma^2 T_{xx} + i\frac{v}{c}\gamma^2 T_{x4}) + \gamma^2 T_{4x} + i\frac{v}{c}\gamma^2 T_{44} & -i\frac{v}{c}\gamma T_{xy} + \gamma T_{4y} & -i\frac{v}{c}\gamma T_{xz} + \gamma T_{4z} & -i\frac{v}{c}(-i\frac{v}{c}\gamma^2 T_{xx} + \gamma^2 T_{x4}) - i\frac{v}{c}\gamma^2 T_{4x} + \gamma^2 T_{44} \end{array}\right)$$

Because $T_{\mu\nu} = T_{\nu\mu}$ is symmetric, we can write this as,

$$T'_{\mu\nu} = \left(\begin{array}{c|c|c|c} \gamma^2(T_{xx} - \frac{v^2}{c^2}T_{44} + 2i\frac{v}{c}T_{x4}) & \gamma T_{xy} + i\frac{v}{c}\gamma T_{4y} & \gamma T_{xz} + i\frac{v}{c}\gamma T_{4z} & \gamma^2(T_{x4} + \frac{v^2}{c^2}T_{x4} - i\frac{v}{c}(T_{xx} - T_{44})) \\ \hline \gamma T_{yx} + i\frac{v}{c}\gamma T_{y4} & T_{yy} & T_{yz} & -i\frac{v}{c}\gamma T_{yx} + \gamma T_{y4} \\ \hline \gamma T_{zx} + i\frac{v}{c}\gamma T_{z4} & T_{zy} & T_{zz} & -i\frac{v}{c}\gamma T_{zx} + \gamma T_{z4} \\ \hline \gamma^2(T_{x4} + \frac{v^2}{c^2}T_{x4} - i\frac{v}{c}(T_{xx} - T_{44})) & -i\frac{v}{c}\gamma T_{xy} + \gamma T_{4y} & -i\frac{v}{c}\gamma T_{xz} + \gamma T_{4z} & \gamma^2(T_{44} - \frac{v^2}{c^2}T_{xx} - 2i\frac{v}{c}T_{x4}) \end{array}\right) \tag{7.6.6}$$

Now using $T_{44} = u$ and $T_{i4} = -ic\Pi_i$, we get,

$$u' = \gamma^2(u - \frac{v^2}{c^2}T_{xx} - 2v\Pi_x), \tag{7.6.7}$$

$$-ic\Pi'_x = \gamma^2(-ic\Pi_x(1 + \frac{v^2}{c^2}) - i\frac{v}{c}(T_{xx} - u)) \tag{7.6.8}$$

$$-ic\Pi'_y = \gamma(-ic\Pi_y - i\frac{v}{c}T_{yx}) \tag{7.6.9}$$

$$-ic\Pi'_z = \gamma(-ic\Pi_z - i\frac{v}{c}T_{zx}) \tag{7.6.10}$$

Using $d^3r' = (1/\gamma)d^3r$ (since by the FitzGerald contraction $dx' = dx/\gamma$, while $dy' = dy$ and $dz' = dz$), the total energy U and total momentum $\boldsymbol{\mathcal{P}}$ transform like,

$$U' = \int d^3r'\, u' = \gamma \int d^3r\, (u - \frac{v^2}{c^2}T_{xx} - 2vp_x) \tag{7.6.11}$$

$$\mathcal{P}'_x = \int d^3r'\, \Pi'_x = \gamma \int d^3r\, (\Pi_x(1 + \frac{v^2}{c^2}) + \frac{v}{c^2}(T_{xx} - u)) \tag{7.6.12}$$

$$\mathcal{P}'_y = \int d^3r'\, \Pi'_y = \int d^3r\, (\Pi_y + \frac{v}{c^2}T_{yx}) \tag{7.6.13}$$

$$\mathcal{P}'_z = \int d^3r'\, \Pi'_z = \int d^3r\, (\Pi_z + \frac{v}{c^2}T_{zx}) \tag{7.6.14}$$

If the frame $\mathcal{K}$ is the rest frame of the charged particle, then $\boldsymbol{\mathcal{P}} = 0$. Also, by symmetry we expect $\int d^3r\, T_{ij} = 0$ when $i \neq j$. The above then becomes,

$$U' = \int d^3r'\, u' = \gamma \int d^3r\, (u - \frac{v^2}{c^2}T_{xx}) = \gamma U - \gamma\frac{v^2}{c^2} \int d^3r\, T_{xx} \tag{7.6.15}$$

$$\mathcal{P}'_x = \int d^3r'\, \Pi'_x = \gamma\frac{v}{c^2} \int d^3r\, (T_{xx} - u) = -\gamma\frac{v}{c^2}U + \gamma\frac{v}{c^2} \int d^3r\, T_{xx} \tag{7.6.16}$$

$$\mathcal{P}'_y = 0 \tag{7.6.17}$$

$$\mathcal{P}'_z = 0 \tag{7.6.18}$$

Now if $\boldsymbol{\mathcal{P}}$ and U were an energy-momentum 4-vector $(\boldsymbol{\mathcal{P}}, iU/c)$, it would transform like,

$$U' = \gamma U - v\gamma\mathcal{P}_x = \gamma U \qquad (\text{since}, \mathcal{P}_x = 0\,\text{in frame}\,\mathcal{K}) \tag{7.6.19}$$

$$\mathcal{P}'_x = \gamma\mathcal{P}_x - \gamma\frac{v^2}{c^2}U = -\gamma\frac{v}{c^2}U \tag{7.6.20}$$

So we see that the total electromagnetic energy U and momentum $\boldsymbol{\mathcal{P}}$ will transform like a 4-vector going from $\mathcal{K}$ to $\mathcal{K}'$ only if $\int d^3r\, T_{xx} = 0$.

However, if $T_{\mu\nu}$ is the the stress tensor of the electromagnetic fields only, then we know that in the rest frame $\mathcal{K}$ of the particle, $T_{xx} = E_x^2 + B_x^2 - \frac{1}{2}\left[E^2 + B^2\right] \neq 0$. Moreover, as we show below, $\int d^3r\, T_{xx} \neq 0$. So there would seem to be a problem ...

The 4/3 Problem

Another well known difficulty arises if we try to interpret the mass of the charge as being purely electromagnetic in nature, i.e. due to the energy contained in the charge's electromagnetic fields, as we considered in one of the Problem Sets.

Consider our result of Eq. (7.6.16) for the Lorentz transformation from the reference frame $\mathcal{K}$ to the frame $\mathcal{K}'$, where $\mathcal{K}$ is the frame in which the charge is instantaneously at rest with $\mathbf{v} = 0$.

$$\mathcal{P}'_x = -\gamma \frac{v}{c^2}\left[U - \int d^3r\, T_{xx}\right] \tag{7.6.21}$$

Let us also assume that the charge is not accelerating, so that in $\mathcal{K}$ we also have $\mathbf{a} = 0$. The primed frame is then the frame in which the charge moves with constant velocity $-v\hat{\mathbf{x}}$.

In the rest frame, since $\mathbf{a} = 0$, the charge is a static charge. The electromagnetic fields are spherically symmetric, so

$$\int d^3r\, T_{xx} = \int d^3r\, T_{yy} = \int d^3r\, T_{zz} \tag{7.6.22}$$

Also, the off-diagonal elements vanish, $\int d^3r\, T_{ij} = 0$ for $i \neq j$. Since the stress tensor $T_{\mu\nu}$ has zero trace, $T_{\mu\mu} = 0$, we get

$$\int d^3r\,[u + T_{xx} + T_{yy} + T_{zz}] = \int d^3r\,[u + 3T_{xx}] = 0 \quad \Rightarrow \quad \int d^3r\, T_{xx} = -\frac{1}{3}\int d^3r\, u = -\frac{1}{3}U \tag{7.6.23}$$

So we then get

$$\mathcal{P}'_x = -\gamma\frac{v}{c^2}\left[U - \int d^3r\, T_{xx}\right] = -\gamma\frac{v}{c^2}\left[U + \frac{1}{3}U\right] = -\gamma\frac{v}{c^2}\frac{4}{3}U \tag{7.6.24}$$

For a slow moving particle, $v \to 0$, one has $\gamma \to 1$ and so,

$$\mathcal{P}'_x = -\frac{4}{3}\frac{U}{c^2}v. \tag{7.6.25}$$

However, if the mass of the charge was purely electromagnetic in nature, i.e. the rest mass $m = U/c^2$, we would expect to have found, $\mathcal{P}'_x = -mv = -(U/c^2)v$, without the factor of 4/3. This is known as the "4/3 problem." The source of this problem is that $\int d^3r\, T_{xx} \neq 0$ in the rest frame of the charge.

The Poincaré Stress

The Poincaré stress is an idea that says there must be mechanical stress on the charged particle to keep the charge bound onto the particle (rather than repelling and spreading out to infinity). We already mentioned the need for such stresses at the end of Unit 4-2, when we discussed the outward force on the surface on a spherical shell model for an electron. A charged particle with such mechanical stresses added to the electromagnetic stress would have a total stress with $\int d^3r\, T_{xx}^{\text{total}} = 0$, so that it is stable. If one then puts this $\int d^3r\, T_{xx}^{\text{total}} = 0$ into the formula for P'_x, then one gets the expected $P'_x = -(U/c^2)v$. Thus we cannot consider just electromagnetic stress when talking about a charged particle, we must also include the mechanical stress that holds the charge on the particle.

More generally, when considering the total energy and momentum of a charged particle in its instantaneous rest frame (even if $\mathbf{a} \neq 0$ in that frame), we cannot consider only the stress tensor $T_{\mu\nu}$ due to the electromagnetic fields, but rather we must include the contribution from the Poincare stresses. These should be such that $\int d^3r\, T_{xx}^{\text{total}} = 0$ if the charged particle is stable, i.e. there is not a net repulsive force that wants to make the charge explode outward. In that case then the energy-momentum components of $T_{\mu\nu}^{\text{total}}$ will transform like a 4-vector, as we assumed in deriving the relativistic version of Larmor's formula.