Then we get

$$n = \frac{N}{V} = \frac{n(0)}{V} + \frac{g_{3/2}(z)}{\lambda^3}$$

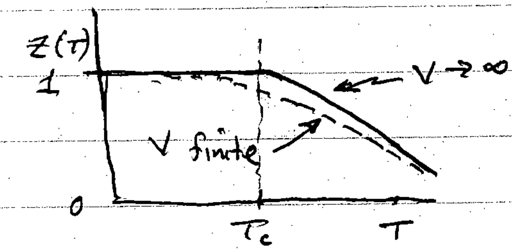
$$n = n_0 + \frac{g_{3/2}(z)}{\lambda^3} \qquad \text{where } n_0 = \frac{n(0)}{V} \text{ density of bosons in ground state}$$

For a system with fixed n, at higher T one can always choose z so that $n = \frac{g_{3/2}(z)}{\lambda^3}$ and $n_0 = 0$.

But when $T < T_c$ it is necessary to have $n_0 > 0$.

Using $n(0) = \frac{z}{1-z}$ we can write above as

$$n = \frac{z}{1-z}\frac{1}{V} + \frac{g_{3/2}(z)}{\lambda^3}$$



For $T > T_c$ we will have a solution to the above for some fixed $z < 1$. In thermodynamic limit $V \to \infty$, the first term will then vanish, ie the <u>density</u> of bosons in the ground state vanishes.

As $T \to T_c$, $z \to 1$ and $\overbrace{\text{the first term}}^{\text{as } V \to \infty}$ $\left(\frac{z}{1-z}\right)\left(\frac{1}{V}\right)$ stays finite to give the additional needed density at $T < T_c$:

$\left(\frac{z}{1-z}\right)$: diverges as $z \to 1$; $\left(\frac{1}{V}\right)$: vanishes as $V \to \infty$

$$\frac{z}{1-z}\frac{1}{V} = n_0 = n - \frac{g_{3/2}(1)}{\lambda^3}$$

T_c defines the Bose-Einstein transition temperature below which the system develops a finite density of particles in the ground state n_0.

n_0 is also called the condensate density.

The particles in the ground state are called the condensate

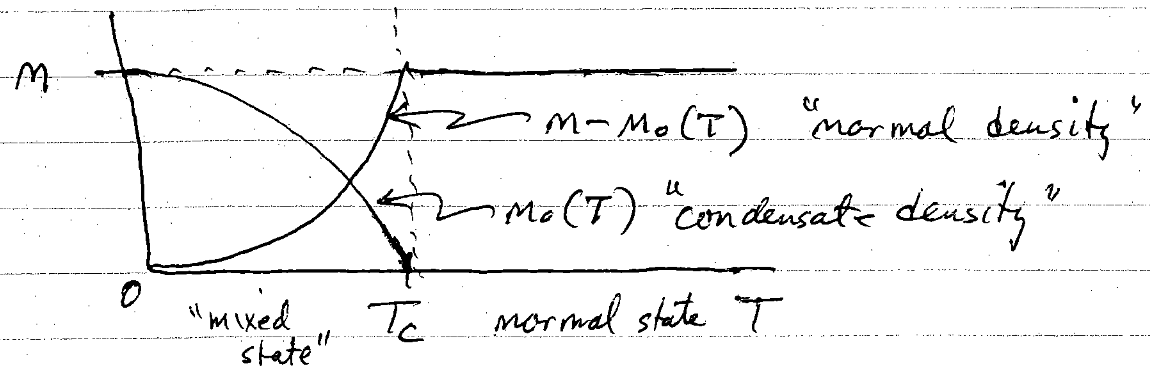
$$\left.\begin{array}{l} z(T) \to 1 \\ \mu(T) \to 0 \end{array}\right\} \text{as } T \to T_c \quad , \quad \left.\begin{array}{l} z(T) = 1 \\ \mu(T) = 0 \end{array}\right\} \text{ for } T \le T_c$$

For $T \le T_c$

$$n_0(T) = n - \frac{g_{3/2}(1)}{\lambda^3} = n - 2.612 \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

$$\boxed{n_0(T) = n\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)}$$

condensate density vanishes continuously as $T \to T_c$ from below



At $T = 0$, all bosons are in condensate

At $T > T_c$, all bosons are in the "normal state"

At $0 < T < T_c$, a macroscopic fraction of bosons are in the condensate, while the remaining fraction are in the normal state — call it the "mixed state"

pressure – separate out ground state from sum as we saw we needed to do in computing N/V

$$\frac{p}{k_BT} = \frac{1}{V}\ln \mathcal{Z} = -\frac{1}{V}\sum_{\vec{k}} \ln\left(1 - z e^{-\beta\epsilon(\vec{k})}\right)$$

$$\simeq \underbrace{-\frac{1}{V}\ln(1-z)}_{\vec{k}=0 \text{ ground state}} \underbrace{- \frac{4\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 \ln\left(1 - z e^{-\beta\hbar^2k^2/2m}\right)}_{\text{all other } |\vec{k}|>0 \text{ states}}$$

$$= \frac{1}{V}\ln\left(\frac{1}{1-z}\right) + \frac{g_{5/2}(z)}{\lambda^3} \qquad \lambda = \left(\frac{h^2}{2\pi m k_BT}\right)^{1/2}$$

where $g_{5/2}(z) \equiv \frac{1}{\Gamma(5/2)}\int_0^\infty dy \frac{y^{3/2}}{z^{-1}e^y - 1}$ as derived when we began our discussion of quantum gases

also recall the number of bosons occupying the ground state is

$$n(0) = \frac{1}{z^{-1}e^{\beta\epsilon(0)} - 1} = \frac{1}{z^{-1}-1} = \frac{z}{1-z}$$

So $n(0)+1 = \frac{z}{1-z} + 1 = \frac{1}{1-z}$

$$\frac{p}{k_BT} = \frac{\ln(n(0)+1)}{V} + \frac{g_{5/2}(z)}{\lambda^3}$$

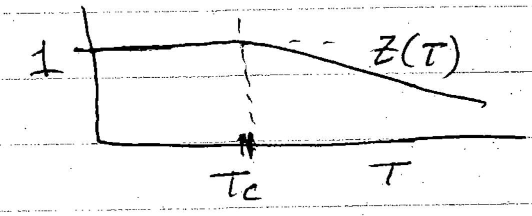
In the thermodynamic limit of $V\to\infty$, the first term always vanishes as $n(0) \leq N = nV$ and $\lim_{V\to\infty}\left[\frac{\ln(nV)}{V}\right] = 0$

So the condensate does not contribute to the pressure. This is not surprising as particles in the condensate have $\vec{k}=0$ and hence carry no momentum. In the kinetic theory of gases, one sees that pressure arises from particles with finite momentum $|\vec{p}|>0$ hitting the walls of the container

So $\frac{p}{k_B T} = \frac{g_{5/2}(z)}{\lambda^3} = g_{5/2}(z)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$

$$p = g_{5/2}(z(T))\left(\frac{2\pi m}{h^2}\right)^{3/2}(k_B T)^{5/2}$$ ← equation of state

for a system of fixed density n, z must be chosen to be a function of T that gives the desired density n.

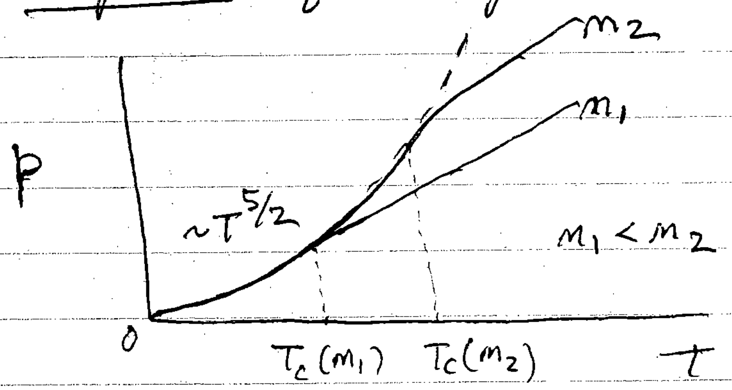


Note $g_{5/2}(z=1) = \zeta(5/2) = 1.342$ is finite

In thermodynamic limit of $V \to \infty$, $z = 1$ for $T \leq T_c(n)$ ← critical temperature depends on the system's fixed density

$$\Rightarrow p = g_{5/2}(1)\left(\frac{2\pi m}{h^2}\right)^{3/2}(k_B T)^{5/2} \quad \text{for } T \leq T_c$$

Note: for $T \leq T_c$, the pressure $p \propto T^{5/2}$ is <u>independent</u> of the system density!



p vs T curves at constant density n

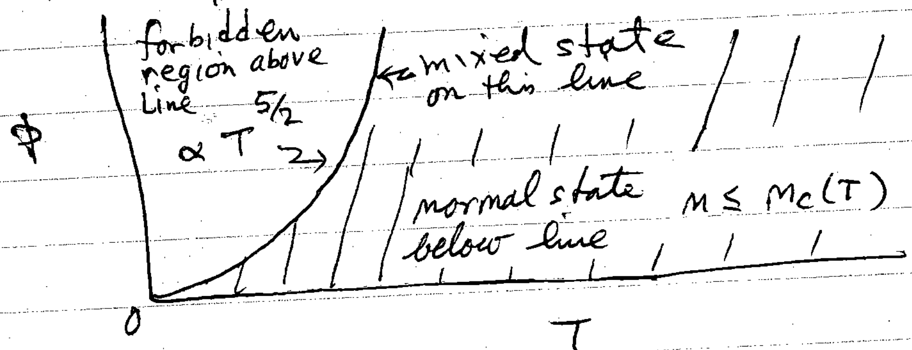
recall $T_c(n) \sim n^{2/3}$

$$T_c(n) = \left(\frac{n}{2.16}\right)^{2/3}\frac{h^2}{2\pi m k_B}$$

Define $n_c(T) = 2.612 \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$ inverse of $T_c(n)$

$n_c(T)$ is the critical density at a given T
— a system with $n > n_c(T)$ will be in a bose condensed mixed state at temperature T.

phase diagram in $p-T$ plane



Can also consider the transition in terms of p and $v = \frac{V}{N} = \frac{1}{n}$ for various fixed T.

At the transition $p \propto T_c(n)^{5/2}$, $T_c(n) \propto n^{2/3}$

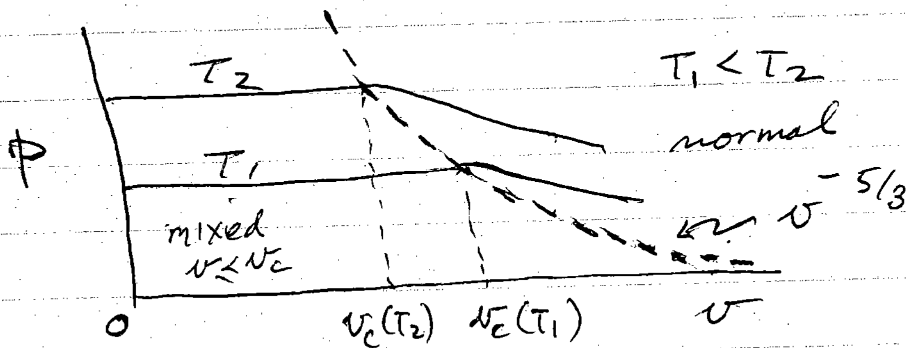
$\Rightarrow$ at the transition $p \propto (n^{2/3})^{5/2} = n^{5/3} = v^{-5/3}$

below the transition p is independent of density and hence independent of v.

For fixed T, the transition occurs when density n exceeds $n_c(T)$, or when v drops below $v_c(T) = \frac{1}{n_c(}$

$v_c(T) \sim T^{-3/2}$

curves of p vs v at constant T



Thermodynamic functions

Earlier we found $\frac{E}{V} = \frac{3}{2} p$

$$\Rightarrow \frac{E}{N} = \frac{3}{2} p \frac{V}{N} = \frac{3}{2} p v = \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(z)$$

$z = 1$ in mixed state
$z < 1$ in normal state

In above we regard $\frac{E}{N}$ as a function of either v or z. That is we either determine v for a given z, T or we determine z needed for a given v, T (Recall $z = e^{\beta\mu}$, $v = \frac{V}{N}$ and N and μ are conjugate variables)

specific heat

$$\frac{C_V}{N k_B} = \frac{1}{k_B}\left(\frac{\partial (E/N)}{\partial T}\right)_{V,N} = \frac{3}{2} v \left\{ \frac{d}{dT}\left(\frac{T}{\lambda^3}\right) g_{5/2}(z) + \frac{T}{\lambda^3} \frac{\partial g_{5/2}(z)}{\partial z} \frac{dz}{dT} \right.$$

For $T \leq T_c$, $z = 1$ so $\frac{dz}{dT} = 0$ and only 1st term remains

$\frac{T}{\lambda^3} \propto T^{5/2}$ so $\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{5}{2}\left(\frac{T}{\lambda^3}\right)\frac{1}{T} = \frac{5}{2}\frac{1}{\lambda^3}$

$z=1$ here for all $T \leq T_c$

$$\Rightarrow \frac{C_V}{Nk_B} = \frac{3}{2} v \left(\frac{5}{2}\frac{1}{\lambda^3}\right) g_{5/2}(1) = \frac{15}{4} g_{5/2}(1) \frac{v}{\lambda^3}$$

$$= \frac{15}{4} g_{5/2}(1)\, v \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

Note, at T_c, $n = \frac{g_{3/2}(1)}{\lambda_c^3}$, and $v = \frac{1}{n}$

$$\frac{C_V(T_c^-)}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{15}{4}\frac{1.341}{2.612} = 1.925$$

this is larger than the classical ideal gas value of $\frac{3}{2}$

So $$\boxed{\frac{C_V}{Nk_B} = 1.925 \left(\frac{T}{T_c}\right)^{3/2} \qquad T \leq T_c}$$

For $T \geq T_c$, z varies with T and we need to evaluate the 2nd term as well

here z depends on T for $T > T_c$

1st term gives $\frac{15}{4} g_{5/2}(z(T)) \frac{v}{\lambda^3}$

2nd term: from Pathria Appendix D Eq(10),

$$z \frac{d}{dz}\left[g_\nu(z)\right] = g_{\nu-1}(z)$$

$$\Rightarrow \frac{d g_{5/2}}{dz}\frac{dz}{dT} = g_{3/2}\frac{1}{z}\frac{dz}{dT}$$

To find $\frac{1}{z}\frac{dz}{dT}$ consider our earlier result for the density when $T > T_c$:

$$n = \frac{g_{3/2}(z)}{\lambda^3} \quad \leftarrow \text{determines } z(T) \text{ for fixed } n$$

for n fixed $\Rightarrow$ $0 = \frac{dn}{dT} = \frac{d}{dT}\left(\frac{1}{\lambda^3}\right) g_{3/2} + \frac{1}{\lambda^3}\frac{dg_{3/2}}{dz}\frac{dz}{dT}$

$$0 = \frac{3}{2}\frac{1}{\lambda^3 T} g_{3/2} + \frac{1}{\lambda^3} g_{1/2} \frac{1}{z}\frac{dz}{dT}$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dT} = -\frac{3}{2}\frac{g_{3/2}}{g_{1/2}}\frac{1}{T}$$

$$\frac{C_V}{Nk_B} = \frac{15}{4} g_{5/2}(z)\frac{v}{\lambda^3} + \frac{3}{2} v \frac{T}{\lambda^3} g_{3/2}(z)\left(-\frac{3}{2}\right)\frac{g_{3/2}(z)}{g_{1/2}(z)}\frac{1}{T}$$

use $n = \frac{1}{v} = \frac{g_{3/2}(z)}{\lambda^3} \Rightarrow \frac{v}{\lambda^3} = \frac{1}{g_{3/2}(z)}$

$$\boxed{\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} \qquad T > T_c}$$

Note $g_{1/2}(1) = \sum_{\ell=1}^{\infty}\frac{1}{\ell^{1/2}} \rightarrow \infty$

So as $T \rightarrow T_c^+$ from above, and $z \rightarrow 1$

$$\frac{C_V(T_c^+)}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{9}{4}\frac{g_{3/2}(1)}{\infty} = \frac{15}{4}\frac{1.341}{2.612} = 1.925$$

$\Rightarrow$ $\boxed{C_V \text{ is continuous at } T_c}$

Finally we want to show that although C_V is continuous at T_c, $\frac{dC_V}{dT}$ is <u>discontinuous</u>

<u>For $T \leq T_c$</u>

$$\frac{C_V}{Nk_B} = 1.925\left(\frac{T}{T_c}\right)^{3/2}$$

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{3}{2}(1.925)\left(\frac{T}{T_c}\right)^{1/2}\frac{1}{T_c} = 2.89\left(\frac{T}{T_c}\right)^{1/2}\frac{1}{T_c}$$

so slope at T_c^- (just below T_c)

is: $$\boxed{\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{2.89}{T_c}, \quad T = T_c^-}$$

<u>For $T > T_c$</u>

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{15}{4}\frac{g_{3/2}\frac{dg_{5/2}}{dz}\frac{dz}{dT} - g_{5/2}\frac{dg_{3/2}}{dz}\frac{dz}{dT}}{(g_{3/2}(z))^2}$$

$$-\frac{9}{4}\frac{g_{1/2}\frac{dg_{3/2}}{dz}\frac{dz}{dT} - g_{3/2}\frac{dg_{1/2}}{dz}\frac{dz}{dT}}{(g_{1/2})^2}$$

$$= \frac{1}{z}\frac{dz}{dT}\left\{\frac{15}{4}\left(\frac{g_{3/2}^2 - g_{5/2}g_{1/2}}{g_{3/2}^2}\right) - \frac{9}{4}\left(\frac{g_{1/2}^2 - g_{3/2}g_{-1/2}}{g_{1/2}^2}\right)\right\}$$

use $\frac{1}{z}\frac{dz}{dT} = -\frac{3}{2}\frac{g_{3/2}}{g_{1/2}}\frac{1}{T}$ as found earlier

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = -\frac{3}{8T}\frac{g_{3/2}}{g_{1/2}}\left\{15\left(1-\frac{g_{5/2}\,g_{1/2}}{g_{3/2}^2}\right)-9\left(1-\frac{g_{3/2}\,g_{-1/2}}{g_{1/2}^2}\right)\right\}$$

Now as $T \to T_c^+$ from above, $z \to 1$, we have $g_{5/2}(1)$ and $g_{3/2}(1)$ are finite, but $g_{1/2}(1)$ and $g_{-1/2}(1) \longrightarrow \infty$

$\Rightarrow$ at T_c^+

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{45}{8T_c}\frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{27}{8T_c}\,g_{3/2}^2(1)\,\frac{g_{-1/2}(1)}{g_{1/2}^3(1)}$$

Now from Pathria Appendix D Eq (8)

$$g_\nu(1) = \lim_{a\to 0}\frac{\Gamma(1-\nu)}{a^{1-\nu}}$$

So $$\frac{g_{-1/2}(1)}{g_{1/2}^3(1)} = \lim_{a\to 0}\frac{\Gamma(3/2)}{a^{3/2}}\left(\frac{a^{1/2}}{\Gamma(1/2)}\right)^3 = \frac{\Gamma(3/2)}{[\Gamma(1/2)]^3}$$

$$= \frac{\frac{1}{2}\pi^{1/2}}{\pi^{3/2}} = \frac{1}{2\pi} \qquad \text{since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi},\quad \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

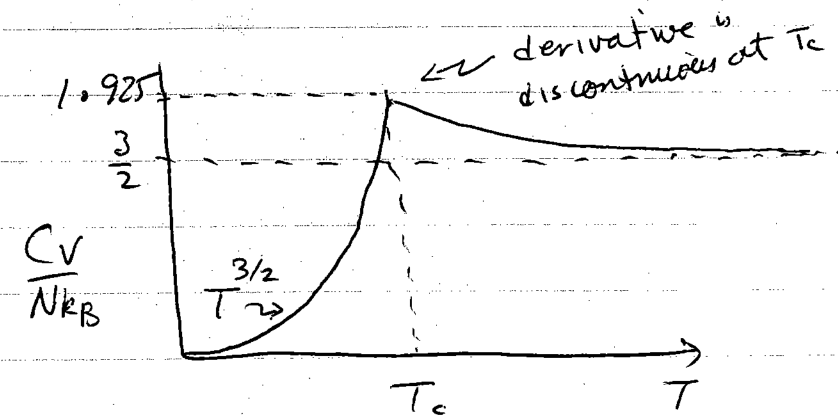
$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{45}{8}\frac{1.341}{2.612}\frac{1}{T_c} - \frac{27}{8}\frac{(2.612)^2}{2\pi}\frac{1}{T_c}$$

$$= \frac{2.89}{T_c} - \frac{3.66}{T_c} = -\frac{0.77}{T_c}$$

$$\boxed{\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{-0.77}{T_c}, \quad T = T_c^+}$$

slope of C_V is discontinuous at T_c.

C_V has a cusp at T_c



goes to classical $\frac{3}{2}$ as $T \to \infty$

$\frac{dC_V}{dT} > 0$ for $T = T_c^-$

$\frac{dC_V}{dT} < 0$ for $T = T_c^+$

Entropy

For single species gas we had for Gibbs free energy

$$G = N\mu$$

Also $G = E - TS + pV$ (since G is Legendre transform from E with respect to S and V)

$$\Rightarrow N\mu = E - TS + pV$$

or $$S = \frac{E + pV - N\mu}{T}$$

$$\frac{S}{Nk_B} = \frac{E + pV}{Nk_BT} - \frac{\mu}{k_BT}$$

we had earlier $E = \frac{3}{2}pV \Rightarrow pV = \frac{2}{3}E$

$$\frac{S}{Nk_B} = \frac{5}{3}\frac{E}{N}\frac{1}{k_BT} - \frac{\mu}{k_BT}$$

$$z = e^{\mu/k_B T}, \quad z = 1 \text{ for } T < T_c$$

we had earlier $\dfrac{E}{N} = \dfrac{3}{2} \dfrac{k_B T \, v}{\lambda^3} g_{5/2}(z)$

and $n = \dfrac{1}{v} = \dfrac{g_{3/2}(z)}{\lambda^3}$ for $T > T_c$

$$\Rightarrow \frac{S}{Nk_B} = \frac{5}{2} \frac{v}{\lambda^3} g_{5/2}(z) - \ln z = \begin{cases} \dfrac{5}{2} \dfrac{g_{5/2}(z)}{g_{3/2}(z)} - \ln z, & T > T_c \\ \dfrac{5}{2} \dfrac{v}{\lambda^3} g_{5/2}(1), & T \leq T_c \end{cases}$$

Note: For $T \leq T_c$ we had ~~that the density of the "normal"~~ a density $n_0 = n - \dfrac{g_{3/2}(1)}{\lambda^3}$ in the condensate, and a density $\dfrac{g_{3/2}(1)}{\lambda^3} \equiv n_n$ in the "normal" state (ie the density of excited particles)

$$\Rightarrow \text{for } T \leq T_c, \quad \frac{S}{Nk_B} = \frac{5}{2}\left(\frac{n_n}{n}\right) \frac{g_{5/2}(1)}{g_{3/2}(1)} \rightarrow 0 \text{ as } T \rightarrow 0$$

we can imagine that each normal particle carries entropy $\dfrac{5}{2} k_B \dfrac{g_{5/2}(1)}{g_{3/2}(1)}$. The entropy/per particle at $T < T_c$ is just the ~~[illegible]~~ above entropy per "normal" particle times the fraction of normal particles.

$\Rightarrow$ normal particles carry the entropy
condensate has zero entropy

entropy difference per particle between normal state and condensed state is $\dfrac{\Delta S}{N} = \dfrac{5}{2} k_B \, g_{5/2}(1)/g_{3/2}(1)$

latent heat of condensation

$$L = T\Delta S = \frac{5}{2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

energy released upon converting one normal particle to one condensate particle.

$\Rightarrow$ mixed phase is like coexistence region of a 1st order phase transition (like water $\Leftrightarrow$ ice – need to remove energy to turn water to ice)

$\Rightarrow$ "Two fluid" model of mixed region

The existence of Bose Einstein condensation is particular to the dimensionality of the system. To see this, consider a general d-dimensional system. Then

$$\frac{1}{V}\sum_k n(\epsilon(k)) \rightarrow \propto \int dk\, k^{d-1} \frac{1}{z^{-1} e^{\beta\hbar^2 k^2/2m} - 1}$$

is largest when $z \rightarrow 1$, so consider this case

$$\propto \int dk\, k^{d-1} \frac{1}{e^{\beta\hbar^2 k^2/2m} - 1} \qquad \text{let } y = \frac{\beta\hbar^2 k^2}{2m}$$

$$k = \sqrt{\frac{2my}{\beta\hbar^2}}$$

$$\propto \left(\frac{2m}{\beta\hbar^2}\right)^{d/2} \int_0^\infty dy \frac{y^{\frac{d}{2}-1}}{e^y - 1} \qquad dk = \sqrt{\frac{2my}{\beta\hbar^2}}\frac{dy}{2y}$$

again the most singular part of the integral is as $y \rightarrow 0$

for $y^* \ll 1$, $$\int_0^{y^*} dy \frac{y^{\frac{d}{2}-1}}{e^y - 1} \approx \int_0^{y^*} dy \frac{y^{\frac{d}{2}-1}}{y} = \int_0 dy\, y^{\frac{d}{2}-2}$$

The integral will converge at its lower limit $y \rightarrow 0$ only for $\frac{d}{2} - 2 > -1$ or $d > 2$

For $d \leq 2$, the integral will diverge. Therefore it will always be possible to find a z such that

$$n = \frac{N}{V} = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{z^{-1} e^{\beta\hbar^2 k^2/2m} - 1}$$

$\Rightarrow$ No Bose Einstein Condensation in two dimensions or below!