Bose-Einstein Condensation in laser cooled gases

gases of alkali atoms Li, Na, K, Rb, Cs

- all have a single s-electron in outermost shell.
  - important for efficiency of laser cooling
- use isotopes such that total intrinsic spin of all electrons and nucleons add up to an integer $\hbar$
  $\Rightarrow$ atoms are bosons
- all have a net magnetic moment - used to confine dilute gas of atoms in a "magnetic trap" $\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$
- use "laser cooling" to get very low temperatures in low density gases, to try and see BEC

magnetic trap $\rightarrow$ effective harmonic potential for atoms

$$V(r) = \tfrac{1}{2} m \omega_0^2 r^2 \qquad \omega_0 \approx 2\pi \times 100\,\text{Hz}$$

1995 - 10^3 atoms with $T_c \sim 100\,\text{nK}$

1999 - 10^8 atoms with $T_c \sim \mu\text{K}$ gas size $\sim$ many microns

How was BEC in these systems observed?

energy levels of ideal (non-interacting) bosons in harmonic trap

$$\epsilon(n_x, n_y, n_z) = (n_x + n_y + n_z + 3/2)\hbar\omega_0$$

n_x, n_y, n_z integers

ground state condensate wavefunction

$$\psi_0(r) \sim e^{-r^2/2a^2} \quad \text{with} \quad a = \left(\frac{\hbar}{m\omega_0}\right)^{1/2}$$

$a \sim 1\,\mu\text{m}$ for current traps

$\Rightarrow$ Condensate has spatial extent $\sim a$

The spatial extent of the n^{th} excited energy level is roughly

$$m\omega_0^2 \langle r^2 \rangle \sim \epsilon(n) \simeq n\hbar\omega_0$$

$$\Rightarrow \langle r^2 \rangle \sim \frac{n\hbar}{m\omega_0} \quad \text{or} \quad \sqrt{\langle r^2 \rangle} = \left(\frac{n\hbar}{m\omega_0}\right)^{1/2}$$

For $k_B T \gg \hbar\omega_0$, the atoms are excited up to level $n \sim \frac{k_B T}{\hbar\omega_0}$

$\Rightarrow$ spatial extent of the normal component of the gas is

$$R \sim \left(\frac{n\hbar}{m\omega_0}\right)^{1/2} \sim \left(\frac{\hbar k_B T}{\hbar m\omega_0^2}\right)^{1/2} = \left(\frac{k_B T}{m\omega_0^2}\right)^{1/2}$$

$$R \sim a\left(\frac{k_B T}{\hbar\omega_0}\right)^{1/2} \gg a$$

If T_c is the BEC transition temperature, then for $T > T_c$ one sees a more or less uniform cloud of atoms with radius $R \sim a\,(k_B T/\hbar\omega_0)^{1/2} \gg a$. But when one cools to $T < T_c$, one now has a finite fraction of the atoms condensed in the ground state. $\rightarrow$ superimposed on the atomic cloud of radius R one sees the growth of a sharp peak in density at the center of cloud – this peak has a radius $a \ll R$.

To find the Bose-Einstein condensation temperature

The number of particles in the system is

$$N = \sum_{n_x, n_y, n_z} \left[\frac{1}{z^{-1} e^{\varepsilon(n_x, n_y, n_z)/k_B T} - 1} \right] \leftarrow \text{Bose occupation function}$$

Let $\varepsilon_0 = \varepsilon(0,0,0) = \frac{3}{2}\hbar\omega_0$ the ground state energy

That the Bose occupation function can not be negative $\Rightarrow z^{-1} e^{\varepsilon_0/k_B T} \geq 1 \Rightarrow z \leq e^{\varepsilon_0/k_B T}$

$\Rightarrow \mu \leq \varepsilon_0$

For the Bose condensed state, z assumes its upper limit, i.e. $\mu = \varepsilon_0$ or $z = e^{\varepsilon_0/k_B T}$ — this gives the greatest possible density in the excited states.

$$\Rightarrow \text{for } T \leq T_c, \quad N = \sum_{n_x, n_y, n_z} \left[\frac{1}{e^{(n_x+n_y+n_z)\hbar\omega_0/k_B T} - 1} \right]$$

$$\Rightarrow N = \underbrace{N_0}_{\substack{\text{number in ground state} \\ n_x = n_y = n_z = 0}} + \underbrace{\int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \left[\frac{1}{e^{(n_x+n_y+n_z)\hbar\omega_0/k_B T} - 1} \right]}_{\text{number in excited states}}$$

number in excited states. Make integral approx from sum using $\Delta n_x = \Delta n_y = \Delta n_z = 1$

$$N = N_0 + \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \left[\frac{1}{e^{(x+y+z)} - 1} \right]$$

$$= N_0 + \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \zeta(3) \qquad \zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

At T_c, $N_0 = 0 \Rightarrow k_B T_c = \hbar\omega_0 \left(\frac{N}{\zeta(3)}\right)^{1/3}$

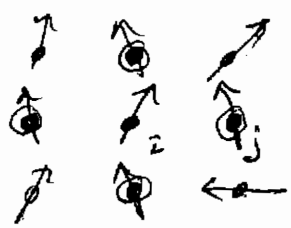
for $T < T_c$, $N_0(T) = N\left(1 - (T/T_c)^3\right)$

power of T/T_c term is different from ideal free gas due to presence of magnetic trapping potential

Classical spin models

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$ simple model of interacting magnetic moments

classical spins $\vec{S}_i$ of unit magnitude $|\vec{S}_i| = 1$ on sites i of a periodic d-dimensional lattice.
$\vec{S}_i$ interacts only with its neighbors $\vec{S}_j$
$\langle ij \rangle$ indicates nearest neighbor bonds of the lattice.
If coupling $J > 0$, then ferromagnetic interaction
ie spins are in lower energy state when they are alligned.



$\vec{S}_i$ interacts with spins on sites labeled by ⊙.

Behavior of model depends significantly on dimensionality of lattice d, and number of components of the spin $\vec{S}$ n.

Examples: $\vec{S} = (S_x, S_y, S_z)$ points in 3-dimensional space
$\underline{n = 3}$ called the Heisenberg model

$\vec{S} = (S_x, S_y)$ restricted to lie in a plane
$\underline{n = 2}$ called the XY model

$S = S_z = \pm 1$ restricted to lie in one direction
$\underline{n = 1}$ called the Ising model

less obvious possibilities $\begin{cases} \underline{n = 0} & \text{called the polymer model} \\ \underline{n = \infty} & \text{called the spherical model} \end{cases}$

We will focus on the Ising model (1925)
$S = \pm 1$

Ensembles

① fixed magnetization $M = \sum_i s_i$ M is total magnetization

partition function $\tilde{Z}(T, M) = \sum_{\substack{\{s_i\} \\ \text{s.t. } \sum_i s_i = M}} e^{-\beta \mathcal{H}[s_i]}$

sum over all spin configurations obeying the constraint $\sum_i s_i = M = N^+ - N^-$

\# up spins   \# down spins

(similar to canonical ensemble with $\sum_i n_i = N$ total # particles)

Helmholtz free energy $F(T, M) = -k_B T \ln \tilde{Z}(T, M)$

② fixed magnetic field

to remove constraint of fixed M we can Legendre transform to a conjugate variable h, ~~the magnetic field~~. We will see that h is just the magnetic field

Gibbs free energy $G(T, h) = F(T, M) - hM$

where $\left(\frac{\partial F}{\partial M}\right)_T = h \Rightarrow \left(\frac{\partial G}{\partial h}\right)_T = -M$

$dF = -SdT + h\,dM$ and $dG = -SdT - M\,dh$

entropy   entropy

To get partition function for G, take Laplace transform of $\tilde{Z}$

$$Z(T,h) = \sum_M e^{\beta h M} \tilde{Z}(T,M)$$

$$= \sum_M e^{\beta h M} \sum_{\substack{\{s_i\} \\ \text{st } \sum_i s_i = M}} e^{-\beta H[s_i]} \qquad \text{use } M = \sum_i s_i$$

looks like interaction of magnetic field h with total magnetization $M = \sum_i s_i$

$$Z(T,h) = \sum_{\{s_i\}} e^{-\beta [H[s_i] - h \sum_i s_i]}$$

↑ unconstrained sum over all spin configs $\{s_i\}$

(similar to grand canonical ensemble with $\sum_i n_i = N$ unconstrained)

$$G(T,h) = -k_B T \ln Z(T,h)$$

Check:

$$\frac{\partial G}{\partial h} = -\frac{k_B T}{Z}\frac{\partial Z}{\partial h} = -\frac{k_B T}{Z}\sum_{\{s_i\}} \frac{\partial}{\partial h}\left(e^{-\beta[H - h\sum_i s_i]}\right)$$

$$= -\frac{k_B T}{Z}\sum_{\{s_i\}} e^{-\beta[H - h\sum_i s_i]}\left(\beta \sum_i s_i\right)$$

$$= -\frac{\sum_{\{s_i\}} e^{-\beta[H - h\sum_i s_i]}\left(\sum_i s_i\right)}{\sum_{\{s_i\}} e^{-\beta[H - h\sum_i s_i]}}$$

$$= -\left\langle \sum_i s_i \right\rangle = -M \qquad \text{so } \frac{\partial G}{\partial h} = -M \text{ as required}$$