

we can work in fixed magnetization or fixed magnetic field ensemble according to our convenience. Usually it is easiest to work with fixed magnetic field. In this case we usually write

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

including the magnetic field part in the definition of H .

$$Z = \sum_{\{S_i\}} e^{-\beta H} \quad \nwarrow \text{includes } h \text{ term}$$

define magnetization density

$$m = \frac{M}{N} = \frac{1}{N} \langle \sum_i S_i \rangle \quad N = \text{total number spins}$$

Helmholtz free energy density: In limit $N \rightarrow \infty$, $F(T, M) = N f(T, m)$

$$\frac{F}{N} \equiv f(T, m) \quad \text{depends on magnetization density}$$

$$df = -s dT + h dm \quad s = \frac{S}{N} \quad \text{entropy per spin}$$

Gibbs free energy density: In limit $N \rightarrow \infty$, $G(T, h) = N g(T, h)$

$$\frac{G}{N} \equiv g(T, h)$$

$$dg = -s dT - m dh$$

$$\left(\frac{\partial f}{\partial m} \right)_T = h \quad , \quad \left(\frac{\partial g}{\partial h} \right)_T = -m$$

What behavior do we expect from Ising model?

For a given h , what is the resulting $m(T, h)$?

For $h > 0$, expect $m > 0$ as energetically favorable for spins to align parallel to h .

For $h < 0$, similarly expect $m < 0$.

In general, $m(T, -h) = -m(T, h)$, since Hamiltonian has the symmetry $H[s_i, h] = H[-s_i, -h]$

What if $h = 0$?

As $T \rightarrow \infty$ we expect each spin to be random so $m \rightarrow 0$.

But even at finite T we might expect $m = 0$ because of symmetry: $H[s_i, 0] = H[-s_i, 0]$ so a configuration $\{s_i\}$ in the partition function sum will enter with the same weight as the configuration $\{-s_i\}$ and so expect $\langle s_i \rangle = 0$.

But at $T = 0$, the system has two degenerate ground states: all up or all down, with $m = \pm 1$. The ground state breaks the symmetry of the Hamiltonian.

More specifically: $\lim_{h \rightarrow 0^+} \lim_{T \rightarrow 0} m(T, h) = +1$

limit $h \rightarrow 0$ from above

limit $h \rightarrow 0$ from below $\rightarrow \lim_{h \rightarrow 0^-} \lim_{T \rightarrow 0} m(T, h) = -1$

Can one have such a broken symmetry state at finite T ?

$$\text{ie } \lim_{h \rightarrow 0^+} m(T, h) = m > 0$$

$$\lim_{h \rightarrow 0^-} m(T, h) = m < 0$$

For a finite size system, N finite, the answer is NO!

For a finite size system, the energy $H[s_i]$ is always finite. The statistical weight of $\{s_i\}$ will always be equal to that of $\{-s_i\}$ in a small h , as we take $h \rightarrow 0$

However, in the thermodynamic limit $N \rightarrow \infty$, the answer can be Yes! Now the energy of states with a finite $\sum s_i$ will grow infinitely large as N . The statistical weight of config $\{s_i\}$ can be infinitely different from that of $\{-s_i\}$ in a small h , even if take $h \rightarrow 0$. ($\infty \times 0 \neq 0$)

$H[s_i] - H[-s_i] \propto hN$ does not necessarily vanish as $h \rightarrow 0$, if $N \rightarrow \infty$ first

It is possible that at finite T

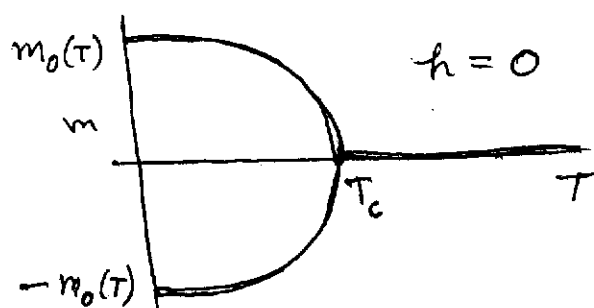
$$\lim_{h \rightarrow 0^+} \left[\lim_{N \rightarrow \infty} m(T, h) \right] = m > 0$$

$$\lim_{h \rightarrow 0^-} \left[\lim_{N \rightarrow \infty} m(T, h) \right] = m < 0$$

It is important to take the limits in the above order - ie first take $N \rightarrow \infty$ in a finite h , and then take $h \rightarrow 0$. Reversing the limits ($h \rightarrow 0$ first, then $N \rightarrow \infty$) gives $m=0$ by symmetry of H .

If such broken symmetry states exist at finite T ,
then do they persist at all T ? or do they disappear
at a well defined T_c ?

Possibility of a phase transition



$$T > T_c, \quad m = 0$$

paramagnetic phase

$$T \leq T_c, \quad m = \pm m_0(T)$$

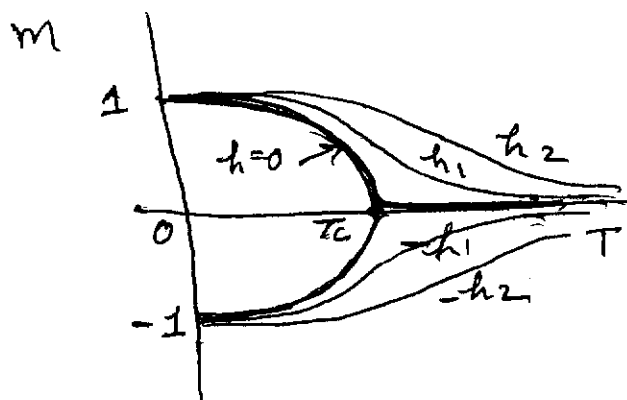
ferromagnetic phase

$m(T, 0)$ is singular at $T = T_c$

T_c is ferromagnetic phase transition

The ordered state at $T \leq T_c$ is a state of
spontaneously broken symmetry. In $h=0$
the system will pick either the up or the
down state to order in, breaking the
symmetry of the Hamiltonian.

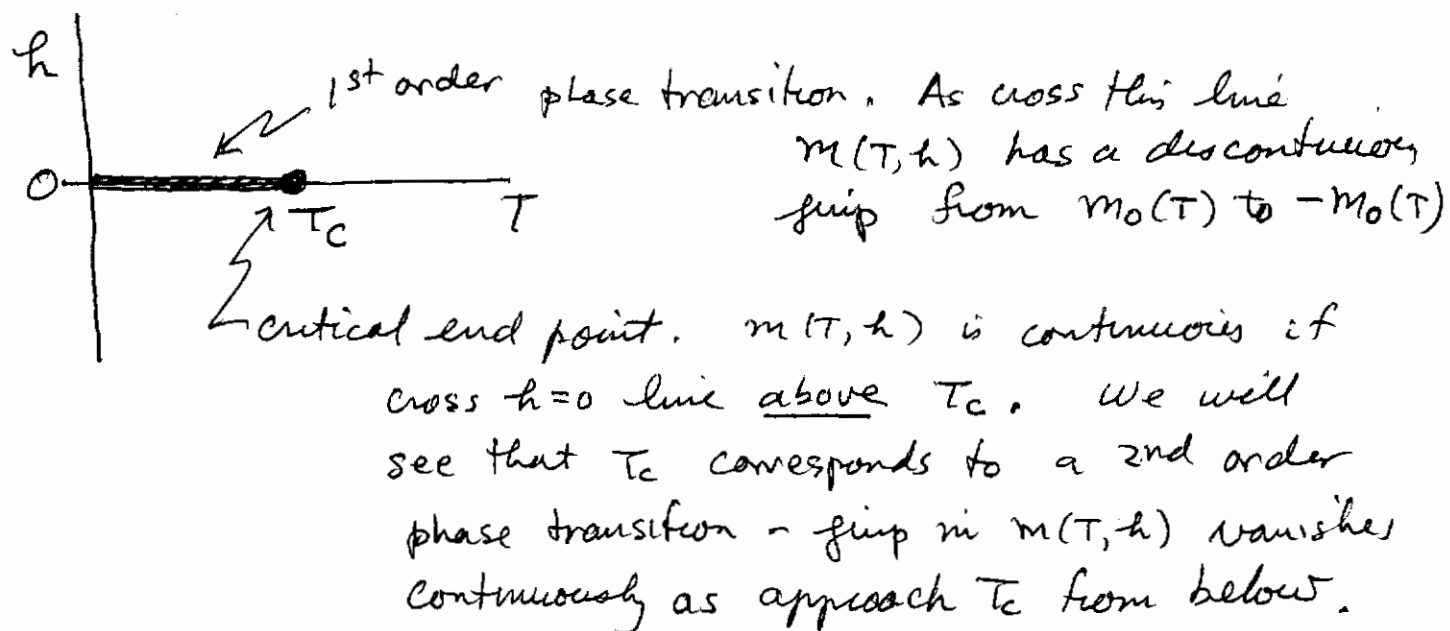
At finite h , expect $m(T, h)$ to behave like



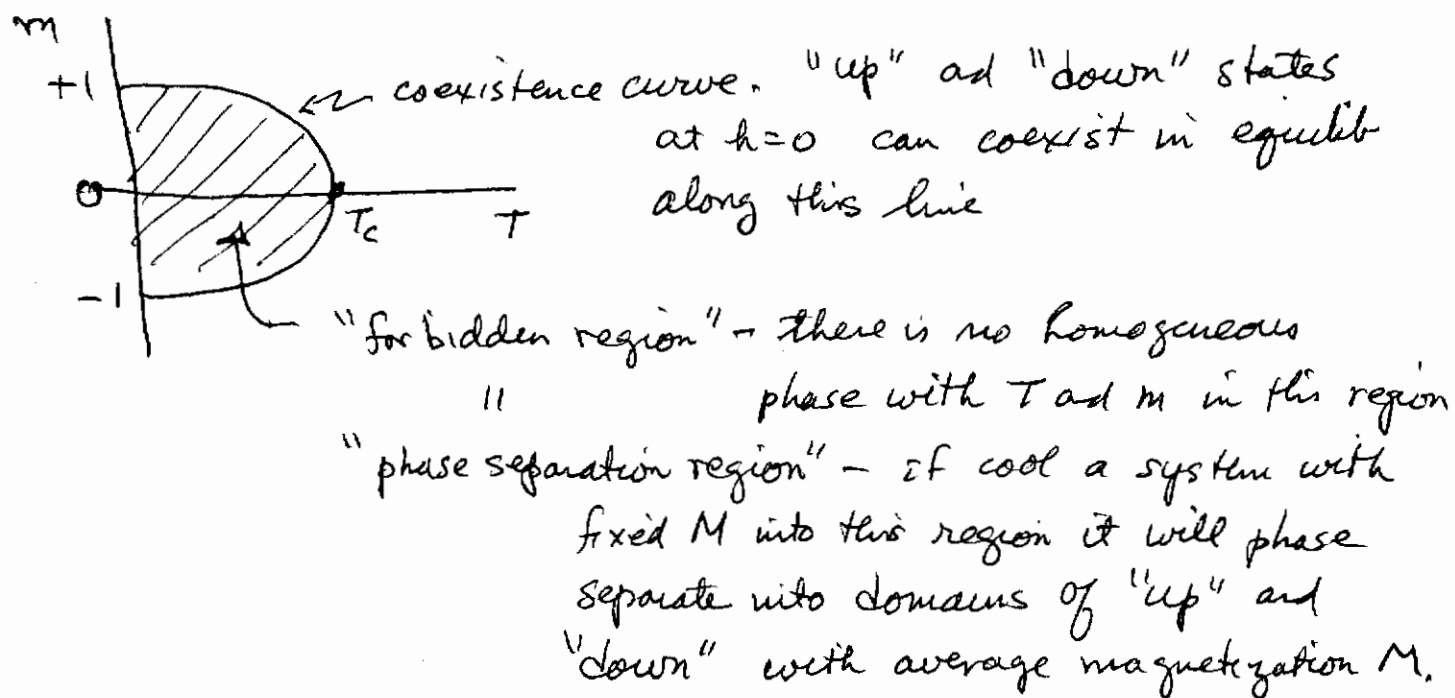
$$h_1 < h_2$$

$m(T, h)$ is smooth
function of T for $h \neq 0$.

Phase diagram in h - T plane



Phase diagram in m - T plane



Many similarities to liquid-gas phase diagram

We said that to have a state of spontaneously broken symmetry at finite T ~~requires~~, one needs to be in thermodynamic limit $N \rightarrow \infty$.

Similarly, true singular phase transitions can only occur in this $N \rightarrow \infty$ limit. Proof as follows:
partition function sum:

$$Z(T, h) = \sum_{\{s_i\}} e^{-\beta H[s_i]}$$

For finite system (N finite) the number of configurations to sum over is 2^N is finite.

Z is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory - has no singularities as vary T, h). As such, Z must itself be an analytic function of T and h .

$\Rightarrow$ Z can have no singularities

$\Rightarrow$ no singularities in any thermodynamic quantities

$\Rightarrow$ no phase transitions.

Only in thermodynamic limit of $N \rightarrow \infty$ is Z now the sum of an infinite ~~many~~ number of analytic functions. Such an infinite sum need NOT be analytic, so phase transitions can exist.