

② At $h=0$ on coexistence line
from (*) with $h=0$ we have

$$\left(1 - \frac{T_c}{T}\right)m + \left[\frac{1}{3}\left(\frac{T_c}{T}\right)^3 + \cancel{\left(\frac{T_c}{T}\right)^2} + \cancel{\left(\frac{T_c}{T}\right)}\right]m^3 = 0$$

as $T \rightarrow T_c^-$, $\left(1 - \frac{T_c}{T}\right) + \frac{1}{3}m^2 = 0$

$$m = \pm \sqrt{\frac{3(T_c - T)}{T}}$$

Define $t = \frac{T_c - T}{T_c}$ $m \propto \pm \sqrt{3t} \propto t^\beta$ $\beta = 1/2$

③ At $h=0$ on coexistence line as $T \rightarrow T_c$
from (**)

$$\frac{\partial h}{\partial m} = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + 3 \left[\left(1 - \frac{T_c}{T}\right) \left(\frac{T_c}{T}\right)^2 + \frac{1}{3} \left(\frac{T_c}{T}\right)^3 \right] m^2 \right\}$$

$$\simeq k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + m^2 \right\} \quad \text{as } T \rightarrow T_c$$

As $T \rightarrow T_c^+$ from above, $m = 0$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T \left(1 - \frac{T_c}{T}\right) = k_B (T - T_c)$$

magnetic susceptibility $\Rightarrow \frac{\partial m}{\partial h} = \chi^+ = \frac{1}{k_B (T - T_c)} \propto \frac{1}{|t|^\gamma}$ $\gamma = 1$

Note: at high temp $T \gg T_c$, $\chi \sim \frac{1}{T}$ just like
in Curie paramagnetism. Hence we say the $T > T_c$
phase is paramagnetic.

As $T \rightarrow T_c^-$ from below, $m^2 = 3 \left(\frac{T_c - T}{T} \right)$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T \left(\left(1 - \frac{T_c}{T} \right) + 3 \left(\frac{T_c - T}{T} \right) \right)$$

$$= 2k_B (T_c - T)$$

$$\frac{\partial m}{\partial h} = \chi^- = \frac{1}{2k_B (T_c - T)} \propto \frac{1}{|t|} \quad \gamma = 1$$

Also $\lim_{T \rightarrow T_c} \left(\frac{\chi^+}{\chi^-} \right) = \frac{2k_B (T_c - T)}{k_B (T - T_c)} = 2 \quad \leftarrow \text{amplitude ratio}$

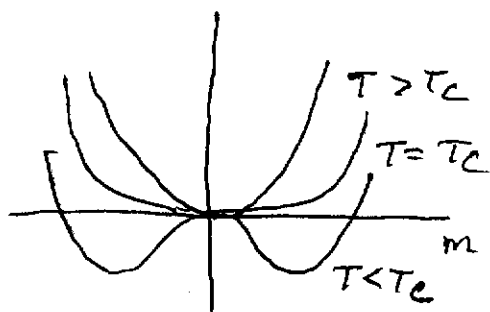
Our results here are identical to what we found for the van der Waal model of liquid-gas transition, if we make the identifications

$$h \leftrightarrow \delta p - \delta p^*(T) \quad \text{distance from coexistence curve}$$

$$m \leftrightarrow \delta v - \delta v_0$$

free energy $f(m, T) - f(0, T) = \int_0^m h(m') dm' \quad \text{use (**) as } T \rightarrow T_c$

$$\Rightarrow f(m, T) - f(0, T) = k_B T \left\{ \frac{1}{2} \left(1 - \frac{T_c}{T} \right) m^2 + \frac{1}{12} m^4 \right\}$$

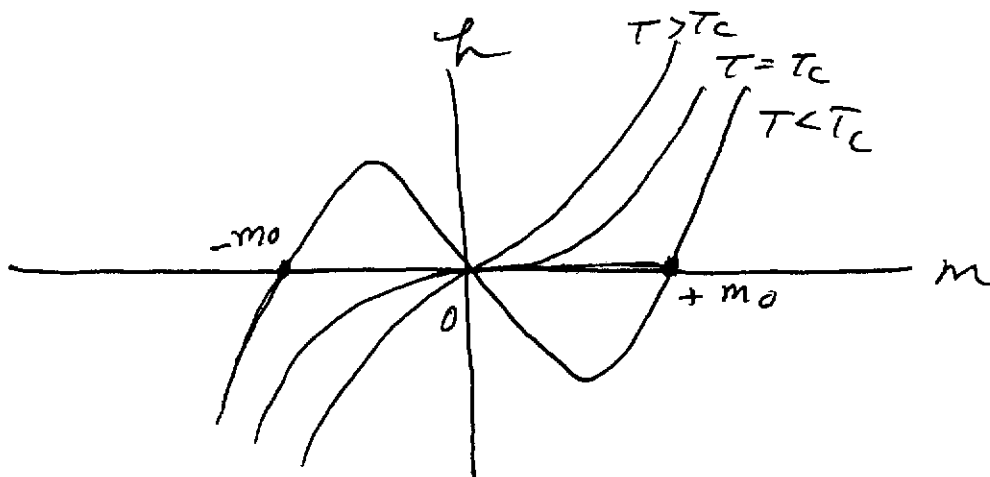


coefficient of m^2 term vanishes at T_c , goes negative below $T_c \Rightarrow$ minimum of $f(m, T)$ changes from $m=0$ to $m = \pm m_0(T)$

$$g(h=0, T) = \min_m f(m, T) \Rightarrow \text{min of } f \text{ gives equilibrium state}$$

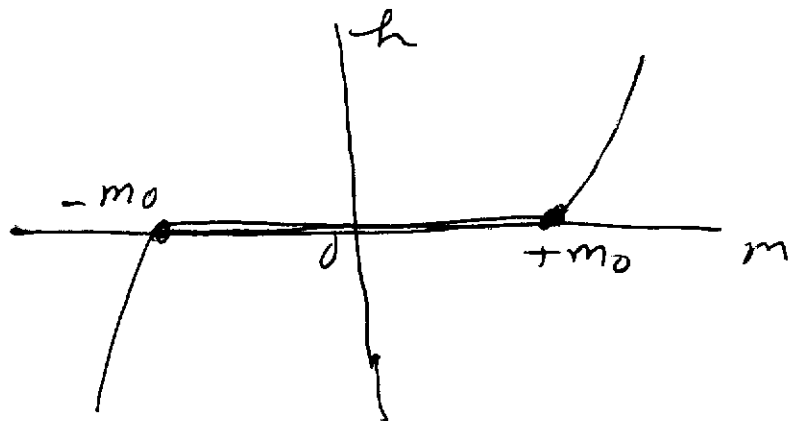
A closer look

$$h = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) m + \frac{1}{3} m^3 \right\}$$



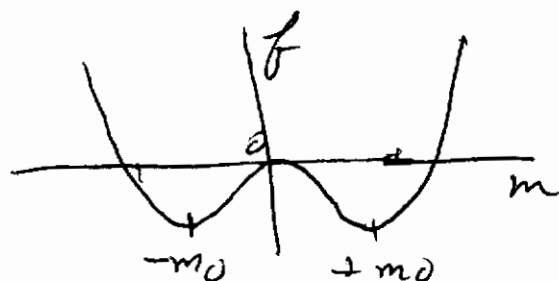
For $T < T_c$ we know that above $h(m)$ curve cannot be valid for $-m_0 \leq m \leq +m_0$.

This is the coexistence region where $h=0$.
For $T < T_c$, the correct $h(m)$ curve is

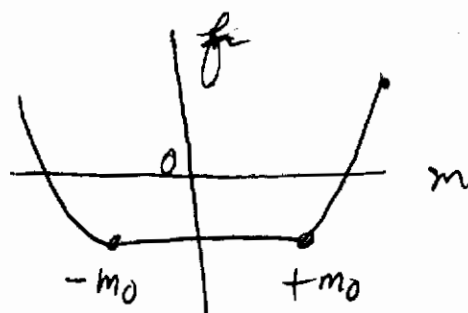


Such a "correction" based on our physical understanding is called the "Maxwell construction" originally done in connection with the van der Waals theory of the liquid to gas phase transition.

If we use the above $h(m)$ for $T < T_c$, ~~then~~
 to compute $f(m, T)$, then instead of



we get

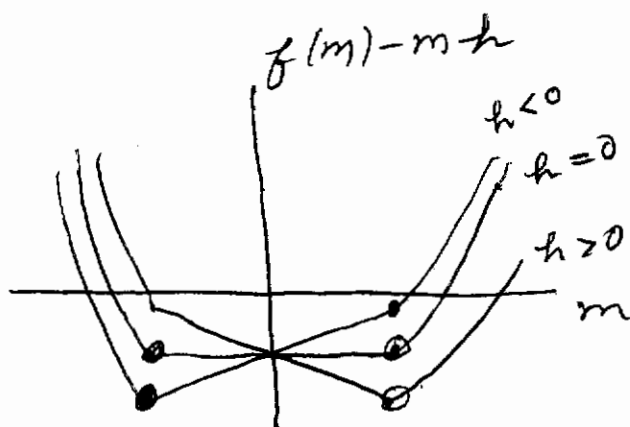


← $f(m)$ with
 Maxwell construction

Note: this can be thought of as if we take the
 top curve and replace it by its convex
 envelop. The top curve cannot be physically
 correct since $f(m)$ must be convex in m .
 Only the lower curve is convex.

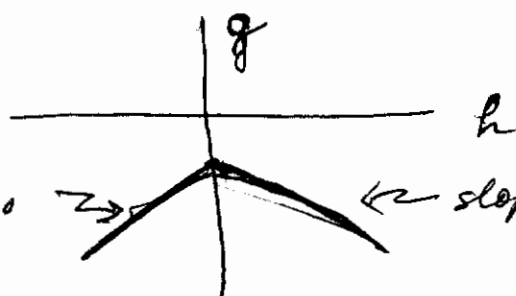
Using the above corrected $f(m)$, we can
 compute

$$g(h, T) = \min_m [f(m, T) - m h]$$



$$T < T_c$$

$$g(h) = \min_m [f(m) - mh] \text{ then results in}$$



$$T < T_c$$

slope is $\omega + m_0$

slope is $-m_0$

$$\frac{dg}{dh} = -m \text{ is discontinuous at } h=0$$

$\Rightarrow g(h)$ has a cusp-like maximum at $h=0$

Note: The mean field approx is exact in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

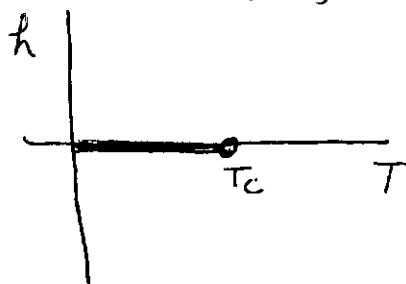
$$\begin{aligned}
 \mathcal{H} &= -\tilde{J} \sum_{i,j} s_i s_j - h \sum_i s_i \\
 &= -\tilde{J} \sum_i s_i \left(\sum_j s_j \right) - h \sum_i s_i \\
 &= -\tilde{J} \sum_i s_i N m - h \sum_i s_i \\
 \mathcal{H} &= -\left(\frac{z}{2} \tilde{J} m + h \right) \sum_i s_i
 \end{aligned}$$

where we took $J \equiv \frac{z}{2} \tilde{J} N$. In infinite range coupling model, need to take coupling $J \propto \frac{1}{N}$ so that total energy scales with $E \propto N$ as desired.

In the above, $m[s_i] = \frac{1}{N} \sum_j s_j$ depends on the config $\{s_i\}$, however it is the same for every spin s_i .

Landau's Theory of phase transitions

Ising model



order parameter $m = \begin{cases} \text{zero} & T > T_c \\ \text{non-zero} & T < T_c \end{cases}$
ordering field h

applying $h \neq 0$ reduces symmetry of Hamiltonian, induces $m \neq 0$.

For ordering field $h=0$, Hamiltonian has higher symmetry. A finite order parameter $m \neq 0$ breaks this symmetry.

For $h=0$, 2nd order phase transition at T_c

Such that $m=0$ for $T > T_c \Rightarrow$ thermodynamic state has full symmetry of $\mathcal{H}$. When $T < T_c$

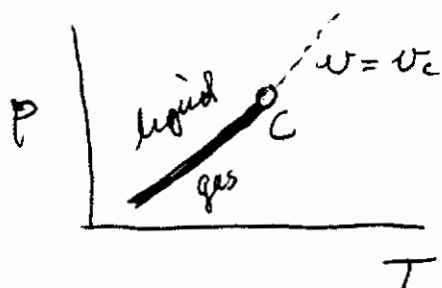
the order parameter becomes finite $m \neq 0 \Rightarrow$ thermodynamic state breaks symmetry of $\mathcal{H}$. Spontaneous symmetry breaking

For $T < T_c$ varying the ordering field h through zero results in a discontinuous jump in the order parameter — 1st order transition line

~~At~~ the 2nd order transition at T_c , that ends the 1st order transition line, m goes to zero continuously as $T \rightarrow T_c^-$.

For liquid-gas transition

(see lecture 26
Spring '05 course)



1st order line does not
have any particular
symmetry with respect
to the natural thermo
variables T, p

crossing the 1st order transition line
there is a discontinuity in the specific
volume $v \equiv \frac{1}{\rho}$ ρ is the particle
density

$\Delta v = v_{\text{gas}} - v_{\text{liq}} \rightarrow 0$ as
approach the critical end point C.

Define v_c = specific volume of system at the
critical point C. Dashed line in phase
diagram is the critical isochore, i.e.
 $v(T, p) = v_c$.

Define order parameter $SV = v - v_c$

Define ordering field Sp as distance from
critical isochore

Landau methodology

- ① Given a physical system with a phase transition, first identify the order parameter m - a quantity that vanishes in the disordered phase and is non-zero in the ordered phase. Often this can be the hard part! Then identify the corresponding ordering field h that couples linearly to m .
- ② Consider a free energy f that depends on T , m , and h such that the Gibbs free energy is given by

$$g(h, T) = \min_m [f(m, T, h)]$$

We can view the function $f(m, T, h)$ in two different ways:

- (i) It is the free energy for fixed T and h when m is ~~not~~ constrained to some value ~~not~~. But the thermodynamic state is when the constraint is removed and m relaxes to the value that minimizes $f(m, T, h)$. The minimizing value of m then determines $g(h, T)$.

or

ii) $f(m, T, h) \equiv \bar{f}(m, T) - mh$
where $\bar{f}(m, T)$ is the usual Helmholtz free energy as function of m and T , and the $-mh$ term is what we add to make the Legendre transformation from $\bar{f}(m, T)$ to $g(h, T)$.

Near the 2nd order critical point where m is small we can expand $f(m, T, h)$ in a Taylor series in m , keeping all terms that are consistent with the symmetries of the physical problem

For the Ising Model

we require by symmetry that

$$f(m, T, h=0) = f(-m, T, h=0)$$

hence we expand

$$f(m, T, h) \approx \underbrace{f_0 + am^2 + bm^4 + \dots}_{\text{no odd powers of } m \text{ here due to symmetry}} - mh \quad \uparrow$$

no odd powers
of m here due
to symmetry

linear coupling
of order parameter
to ordering field

For the liquid-gas transition

we expand

$$f(\delta v, T, \delta p) \approx f_0 + \underbrace{a \delta v^2 + b \delta v^3}_{\substack{\nearrow \\ \text{now no symmetry} \\ \text{to rule out a} \\ \delta v^3 \text{ term,}}} + c \delta v^4 + \dots - \delta v \delta p$$

$\nearrow$
linear coupling
of order parameter
to ordering field

However, we can make a transformation

$$\delta v = \delta v_0 + u$$

where δv_0 is a suitable constant chosen to make the coefficient of u^3 vanish. This will in general result in an additional linear in u term, but that can be absorbed into a redefinition of the ordering field δp .

The result will be

$$f \sim f_0 + \alpha' u^2 + b' u^4 - \delta p' u$$

with exactly the same form as the Ising model.

$\Rightarrow$ Although the liquid-gas transition lacks the symmetry of the Ising model, and although the Ising model & liquid-gas transition look like completely different physical models, we expect them to have the same critical behavior $\Rightarrow$ universality