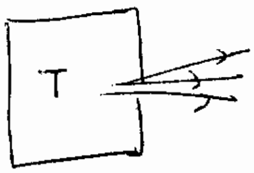
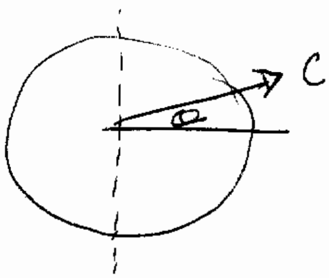
energy flux from a cavity, exiting from a hole



$$\text{flux } \mathcal{F} = \underbrace{\left(\frac{U}{V}\right)}_{\text{energy density}} \underbrace{c}_{\text{speed}} \underbrace{\langle \cos\theta \rangle}_{\text{projection of velocity in outwards direction}}$$



$$\langle \cos\theta \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \sin\theta \cos\theta \quad \leftarrow \text{only in outward direction}$$

$$= \frac{2\pi}{4\pi} \left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2} = \frac{1}{4}$$

$$\mathcal{F} = \left(\frac{U}{V}\right)\frac{c}{4} = \sigma T^4 \quad \leftarrow \text{Stefan Boltzman Law}$$

$$\text{where } \underbrace{\sigma}_{\text{Stefan's constant}} = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} = 5.7 \times 10^{-8} \frac{W}{m^2 \cdot {}^\circ K^4}$$

We also have

$$\frac{pV}{k_B T} = \ln \mathcal{Z} = -\sum_k \overset{\text{polarizations}}{2} \ln\left(1 - e^{-\beta \epsilon_k}\right)$$

$$= -2 \frac{V}{(2\pi)^3} \int dk \, 4\pi k^2 \ln\left(1 - e^{-\beta \hbar c k}\right)$$

$$= -\int_0^\infty d\omega \, g(\omega) \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= -\frac{V}{\pi^2 c^3} \int_0^\infty d\omega \, \omega^2 \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

integrate by parts

$$\frac{pV}{k_B T} = \frac{-V}{\pi^2 c^3}\underbrace{\left[\frac{\omega^3}{3}\ln\left(1-e^{-\beta\hbar\omega}\right)\right]_0^\infty}_{=0} + \frac{V}{\pi^2 c^3}\int d\omega\,\frac{\omega^3}{3}\,\frac{\beta\hbar e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\frac{pV}{k_B T} = \frac{V\beta\hbar}{3\pi^2 c^3}\int_0^\infty d\omega\left(\frac{\omega^3}{e^{\beta\hbar\omega}-1}\right)$$

compare with computation of $\frac{U}{V}$

$$= \frac{\beta}{3}U = \frac{1}{3}\frac{U}{k_B T}$$

$$\Rightarrow \boxed{\frac{1}{3}U = PV}$$ pressure of photon gas

compare to non relativistic ideal gas

$U = \frac{3}{2}Nk_BT$, $PV = Nk_BT \Rightarrow \frac{2}{3}U = PV$

The previous examples of phonons in a solid and Black Body radiation were problems involving bosons with excitation spectrum

$\epsilon = \hbar\omega = \hbar c|\vec{k}|$ (ie linear spectrum)

and zero chemical potential $\mu = 0$.

non-interacting

Now we want to turn to the problem of an ideal quantum gas (bosons or fermions) of physical particles with ~~an excitation~~ an ordinary non-relativistic excitation spectrum

$$\epsilon = \frac{\hbar^2 k^2}{2m} \quad \text{(ie quadratic spectrum)}$$

and $\mu \neq 0$.

Ideal Quantum Gas — Grand canonical ensemble

$$\ln \mathcal{Z} = \pm \sum_i \ln\left(1 \pm e^{-\beta(\epsilon_i - \mu)}\right) \qquad +\text{FD}, \ -\text{BE}$$

for free particles, states can be labeled by ~~wavevector~~ wavevector $\vec{k}$ with $k_\mu = \frac{2\pi n_\mu}{L}$, $n_\mu = 0, \pm 1, \pm 2, \ldots$ due to periodic boundary conditions. volume $V = L^3$

$$\Rightarrow \underset{\text{states}}{\sum_i} \rightarrow \sum_s \sum_{\vec{k}} \rightarrow g_s \frac{V}{(2\pi)^3} \int_0^\infty dk\, 4\pi k^2$$

spin polarizations

spin states for each $\vec{k}$

for free particles, ϵ depends only on $|\vec{k}|$. Define density of states $g(\epsilon)$ such that

$$\frac{g_s}{(2\pi)^3} \int dk\, 4\pi k^2 = \int g(\epsilon)\, d\epsilon$$

$g(\epsilon)$ = # states with energy ϵ per unit energy per volume

$$\Rightarrow g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} k^2 \frac{dk}{d\epsilon}$$

For non-relativistic particles $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$

$$g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} \frac{2m\epsilon}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2\sqrt{\epsilon}}$$

$$= \frac{2\pi g_s}{(2\pi)^3} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon} = \left(\frac{2\pi m}{\hbar^2}\right)^{3/2} \frac{2^{3/2}(2\pi)^{3/2}}{(2\pi)^2} g_s \sqrt{\epsilon}$$

Density of States

$$\boxed{g(\epsilon) = \left(\frac{2\pi m}{\hbar^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \sqrt{\epsilon}}$$

$g \sim \sqrt{\epsilon}$

or $g(\epsilon) = \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \frac{1}{k_B T} \sqrt{\frac{\epsilon}{k_B T}}$ using $\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$

<u>pressure</u>

$$\frac{p}{k_B T} = \frac{1}{V} \ln \mathcal{L} = \pm \frac{1}{V} \sum_i \ln\left(1 \pm z e^{-\beta \epsilon_i}\right) \qquad z = e^{\beta\mu}$$

$$= \pm \int_0^\infty d\epsilon\, g(\epsilon) \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

$$= \pm \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon\, \sqrt{\epsilon} \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

substitute variables $y = \beta\epsilon$

$$\frac{p}{k_B T} = \pm \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty dy\, y^{1/2} \ln\left(1 \pm z e^{-y}\right)$$

integrate by parts $\qquad \lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$ thermal wavelength

$$\frac{p}{k_B T} = \pm \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \left\{ \underbrace{\frac{2}{3} y^{3/2} \ln\left(1 \pm z e^{-y}\right) \Big|_0^\infty}_{=0} - \int_0^\infty dy\, \frac{2}{3} y^{3/2} \frac{(\mp z e^{-y})}{1 \pm z e^{-y}} \right\}$$

$$\boxed{\frac{p}{k_B T} = \frac{4 g_s}{3\sqrt{\pi}\,\lambda^3} \int_0^\infty dy\, \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}}$$

+ FD
− BE

<u>density of particles</u> $\frac{N}{V} = \frac{1}{V} \sum_i \langle n_i \rangle$

$$\frac{N}{V} = \frac{1}{V} \sum_i \frac{1}{z^{-1} e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon\, g(\epsilon) \frac{1}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon\, \frac{\sqrt{\epsilon}}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int dy\, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}$$

$$\boxed{\frac{N}{V} = \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \int_0^\infty dy\, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}}$$

+ FD
− BE

<u>energy density</u> $\quad E = \sum_i \epsilon_i \langle n_i \rangle$

$$\frac{E}{V} = \frac{1}{V} \sum_i \frac{\epsilon_i}{z^{-1} e^{\beta \epsilon_i} \pm 1} = \int_0^\infty d\epsilon \, g(\epsilon) \frac{\epsilon}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \frac{2 g_s}{\sqrt{\pi}\, \lambda^3} k_B T \int_0^\infty dy \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}$$

$$\frac{E}{V} = \frac{3}{2} k_B T \frac{4 g_s}{3\sqrt{\pi}\, \lambda^3} \int_0^\infty \frac{y^{3/2}}{z^{-1} e^{y} \pm 1} = \left(\frac{3}{2} k_B T\right)\left(\frac{p}{k_B T}\right)$$

$$\Rightarrow \frac{E}{V} = \frac{3}{2} p \quad \text{or} \quad \boxed{p = \frac{2}{3} \frac{E}{V}}$$

both fermions <u>and</u> bosons (same result as for classical, ideal gas!!) <u>nonrelativistic only</u>

Define "standard functions" (see Pathria Appendices D and E)

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} + 1} = \sum_{\ell=1}^\infty (-1)^{\ell+1} \frac{z^\ell}{\ell^n}$$

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} - 1} = \sum_{\ell=1}^\infty \frac{z^\ell}{\ell^n}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
$$\Rightarrow \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}$$
$$\Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}$$

In terms of these:

<u>Fermions</u>	<u>Bosons</u>
$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{N}{V} = \frac{g_s}{\lambda^3} f_{3/2}(z)$	$\frac{N}{V} = \frac{g_s}{\lambda^3} g_{3/2}(z)$
$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{E}{N} = \frac{3}{2} k_B T \frac{f_{5/2}(z)}{f_{3/2}(z)}$	$\frac{E}{N} = \frac{3}{2} k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)}$

Equation of state: low densities - virial expansion

$z \ll 1$ "non-degenerate"

← keep lowest terms in series expansion

$$\frac{p}{k_BT} = \frac{g_s}{\lambda^3}\begin{Bmatrix} f_{5/2} \\ g_{5/2}\end{Bmatrix} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right) \qquad \begin{matrix} - \text{ FD} \\ + \text{ BE}\end{matrix}$$

$$\frac{N}{V} = \frac{g_s}{\lambda^3}\begin{Bmatrix} f_{3/2} \\ g_{3/2}\end{Bmatrix} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)$$

$$\Rightarrow \frac{p}{k_BT} = \frac{N}{V}\frac{\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right)}{\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)} = \frac{N}{V}\left(1 \mp \frac{z}{2^{5/2}} + \cdots\right)\left(1 \pm \frac{z}{2^{3/2}} + \cdots\right)$$

$$= \frac{N}{V}\left(1 \pm \frac{z}{2^{3/2}} \mp \frac{z}{2^{5/2}} + \cdots\right) \qquad \frac{1}{2^{3/2}} - \frac{1}{2^{5/2}} = \frac{2}{2^{5/2}} - \frac{1}{2^{5/2}} = \frac{1}{2^{5/2}}$$

$$pV = Nk_BT\left(1 \pm \frac{z}{2^{5/2}} + \cdots\right)$$

↑ quantum correction to classical ideal gas law.

+ FD - p increases compared to classically
- effective repulsion due to Pauli exclusion

− BE - p decreases compared to classically
- effective attraction.

above is similar conclusion to what we saw from 2-particle density matrix.

for small z, the leading term gives $\frac{N}{V} = \frac{g_s}{\lambda^3} z$

or $z = \left(\frac{N}{V}\frac{\lambda^3}{g_s}\right)$ - same result we had classically

$\Rightarrow$ small z limit is the <u>low density limit</u> $n\lambda^3 \ll 1$

$$pV = Nk_BT\left(1 \pm \frac{1}{2^{5/2}g_s}\frac{N}{V}\lambda^3 + \cdots\right) \qquad \cong \text{ high } T$$