As $T \to T_c^-$ from below, $m^2 = 3\left(\frac{T_c - T}{T}\right)$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T\left(\left(1 - \frac{T_c}{T}\right) + 3\left(\frac{T_c - T}{T}\right)\right)$$

$$= 2 k_B (T_c - T)$$

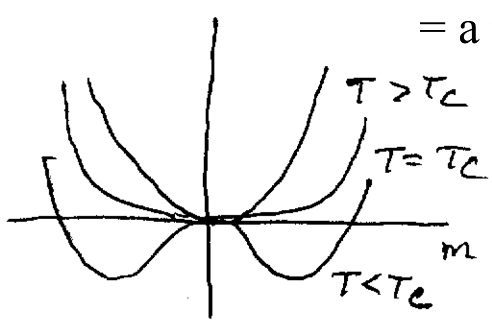
$$\frac{\partial m}{\partial h} = \chi^- = \frac{1}{2 k_B (T_c - T)} \propto \frac{1}{|t|^\gamma} \qquad \gamma = 1$$

Also $\lim_{T \to T_c} \left(\frac{\chi^+}{\chi^-}\right) = \frac{2k_B (T_c - T)}{k_B (T - T_c)} = 2$ ← amplitude ratio

free energy $f(m,T) - f(0,T) = \int_0^m h(m')\, dm'$ use $(**)$ as $T \to T_c$

$$\Rightarrow f(m,T) - f(0,T) = k_B T \left\{ \frac{1}{2}\left(1 - \frac{T_c}{T}\right) m^2 + \frac{1}{12} m^4 \right\}$$

= a m^2 + b m^4



coefficient of m^2 term vanishes at T_c, goes negative below $T_c \Rightarrow$ minimum of $f(m,T)$ changes from $m = 0$ to $m = \pm m_0(T)$

$g(h=0,T) = \min_m f(m,T) \Rightarrow$ min of f gives equilibrium state

④ <u>specific heat at $h=0$</u> along 1st order transition line

from ① we have $m_0^2 = \frac{-a}{2b}$ $T<T_c$, $m_0^2 = 0$ $T>T_c$

$$\Rightarrow \quad g(h=0,T) = f(m_0,T) = f_0(T), \quad T>T_c$$

$$= f_0(T) + a\left(\frac{-a}{2b}\right) + b\left(\frac{-a}{2b}\right)^2, \quad T<T_c$$

$T<T_c$: $$f(m_0,T) = f_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b} = f_0(T) - \frac{a^2}{4b}$$

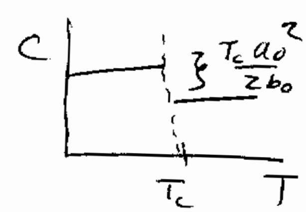
$$= f_0(T) - \frac{a_0^2}{4b_0}(T-T_c)^2 \qquad a = a_0(T_c - T)$$

<u>specific heat</u>

$$s = -\frac{\partial g}{\partial T} \Rightarrow C = T\left(\frac{\partial s}{\partial T}\right)_{h=0} = -T\frac{\partial^2 g}{\partial T^2}$$

$$C = -T\frac{d^2 f(m_0(T),T)}{dT^2}$$

$$= \begin{cases} -T\dfrac{d^2 f_0}{dT^2} & T>T_c \\[2ex] -T\dfrac{d^2 f_0}{dT^2} + \dfrac{T a_0^2}{2b_0} & T<T_c \end{cases}$$



$$\Rightarrow \quad C(T\to T_c^-) - C(T\to T_c^+) = \frac{T_c a_0^2}{2b_0}$$

jump in specific heat at T_c

The piece $\frac{\partial^2 f_0}{\partial T^2}$ is the non singular piece of the specific heat. f_0 is the same as the "reference" free energy we used earlier when integrating the equation of state in the mean field or the van der Waals approx.

We can define a critical exponent α for the specific heat by $C \propto |t|^{-\alpha}$, or

$$\alpha = \lim_{t \to 0} \left[\frac{\ln C}{\ln |t|} \right]$$

For Landau theory this gives $\boxed{\alpha = 0}$

Summary: Landau theory = mean field theory

$h=0, \quad m_0(T) \sim |t|^\beta \quad \beta = 1/2$

$T = T_c, \quad h(m) \propto m^\delta \quad \delta = 3$

$h=0, \quad \chi(T) \propto \frac{1}{|t|^\gamma} \quad \gamma = 1$

$$\lim_{t \to 0} \frac{\chi^+}{\chi^-} = 2$$

$h=0, \quad C(T) \propto |t|^{-\alpha} \quad \alpha = 0$

mean field critical exponents

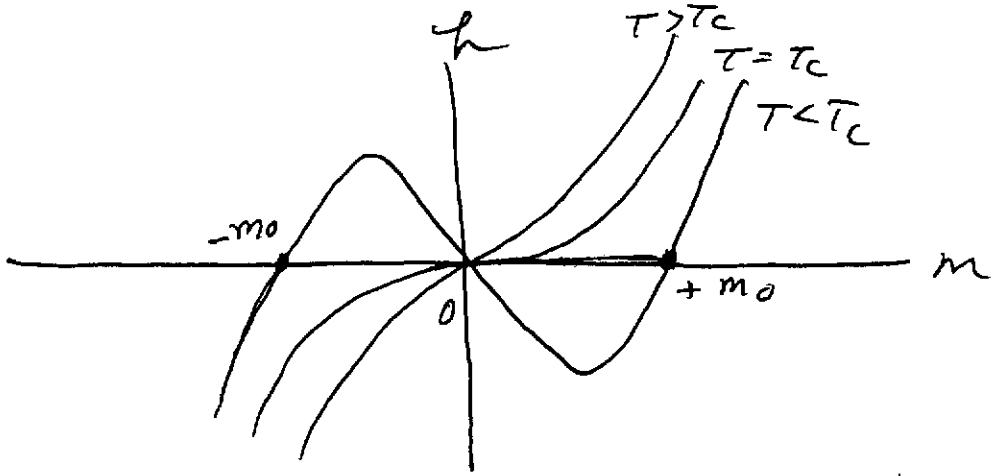
exponent values in mean field approx are indep of dimension d.

From exact solution of 2D Ising model

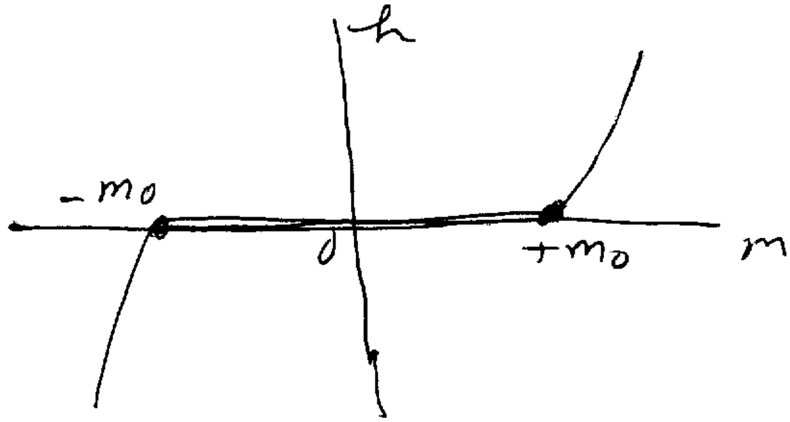
$\delta = 15 \quad \beta = 1/8, \ \gamma = 7/4, \ \underset{\text{log divergence}}{\alpha = 0}, \ \nu = 1 \qquad C \propto \ln|t|$

A closer look

$$h = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) m + \frac{1}{3} m^3 \right\}$$

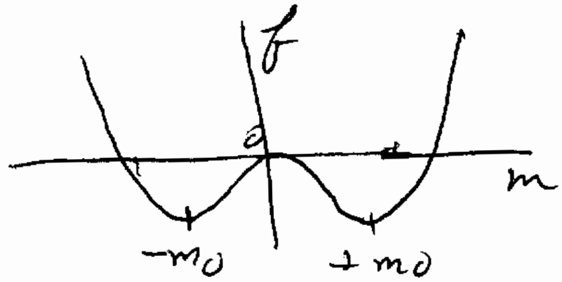


For $T < T_c$ we know that above $h(m)$ curve cannot be valid for $-m_0 \leq m \leq +m_0$
This is the coexistence region where $h = 0$
For $T < T_c$, the correct $h(m)$ curve is

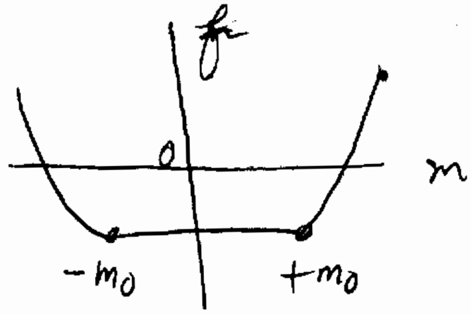


Such a "correction" based on our physical understanding is called the "Maxwell construction" (originally done in connection with the van der Waals theory of the liquid to gas phase transition.

If we use the above $h(m)$ for $T < T_c$, ~~then~~ to compute $f(m,T)$, then instead of



we get

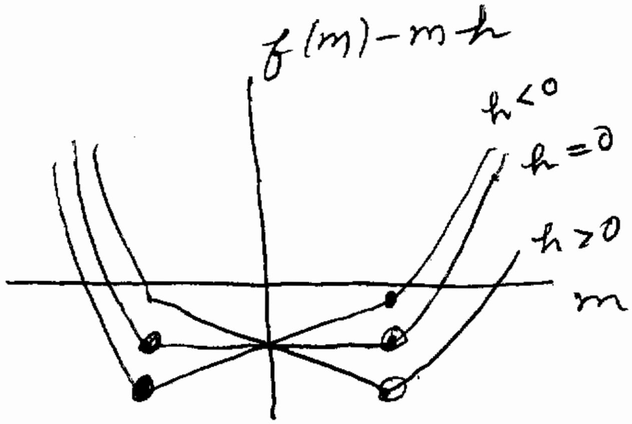


← $f(m)$ with Maxwell construction

Note: this can be thought of as if we take the top curve and replace it by its convex envelope. The top curve cannot be physically correct since $f(m)$ must be convex in m. Only the lower curve is convex.

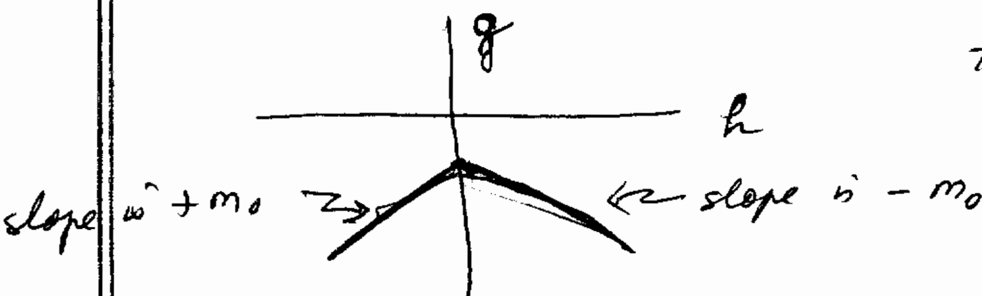
Using the above corrected $f(m)$, we can compute

$$g(h,T) = \min_m \left[f(m,T) - m h \right]$$



$T < T_c$

$g(h) = \min_m \left[f(m) - mh \right]$ then results in



$T < T_c$

$\frac{dg}{dh} = -m$ is discontinuous at $h=0$

$\Rightarrow$ $g(h)$ has a cusp-like maximum at $h=0$

Note: The mean field approx is <u>exact</u> in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

$$\mathcal{H} = -\tilde{J} \sum_{i,j} s_i s_j - h \sum_i s_i$$

$$= -\tilde{J} \sum_i s_i \left(\sum_j s_j \right) - h \sum_i s_i$$

$$= -\tilde{J} \sum_i s_i \, N m - h \sum_i s_i$$

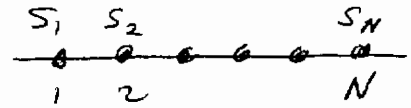
$$\mathcal{H} = -\left(\frac{z J}{2} m + h \right) \sum_i s_i$$

where we took $J \equiv \frac{2}{z} \tilde{J} N$. For infinite range coupling model, need to take coupling $J \propto \frac{1}{N}$ so that total energy scales with $E \propto N$ as desired.

In the above, $m[s_i] = \frac{1}{N} \sum_j s_i$ <u>depends</u> on the config $\{s_i\}$, however it is the same for every spin s_i

Ising model in 1-dimension

$h = 0$ for simplicity



$$\mathcal{H} = -J \sum_{i=1}^{N} S_i S_{i+1}$$

Define $\sigma_i = S_i S_{i+1}$, $i = 1, \cdots, N-1$

$$\sigma_i = \pm 1$$

$$\mathcal{H} = -J \sum_{i=1}^{N-1} \sigma_i \qquad S_1 S_j = \prod_{i=1}^{j-1} \sigma_i = (S_1 S_2)(S_2 S_3) \cdots (S_{j-1} S_j)$$

$$= S_1 S_2^2 S_3^2 \cdots S_{j-1}^2 S_j$$

$$= S_1 S_j$$

For every set of $\{\sigma_i\}_{i=1}^{N-1}$ there are 2 possible spin configurations depending on whether $S_1 = +1$ or -1

For a given value of S_1, then

$$S_j = \frac{1}{S_1} \prod_{i=1}^{j-1} \sigma_i$$

So

$$Z = \sum_{\{S_i\}} e^{\beta J \sum_{i=1}^{N} S_i S_{i+1}} = \underset{\text{two values for } S_1}{2} \sum_{\{\sigma_i\}} e^{\beta J \sum_{j=1}^{N-1} \sigma_j} = 2 \prod_{j=1}^{N-1} \sum_{\sigma_i = \pm 1} e^{\beta J \sigma_i}$$

$$Z = 2 \left[\sum_{\sigma = \pm 1} e^{\beta J \sigma} \right]^{N-1} = 2 \left[2 \cosh \beta J \right]^{N-1}$$

Gibbs free energy

$$G(h=0,T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T(N-1)\ln(2\cosh\beta J)$$

$$g = \lim_{N\to\infty} \frac{G}{N} = -k_B T \ln(2\cosh\beta J)$$

entropy $s = -\left(\frac{\partial g}{\partial T}\right)_{h=0}$ specific heat at const $h=0$ $C = T\left(\frac{\partial s}{\partial T}\right)_{h=0}$

$$= -T\left(\frac{\partial^2 g}{\partial T^2}\right)$$

$$s = k_B \ln(2\cosh\beta J) + \frac{k_B T}{2\cosh(\beta J)} 2 \frac{\partial}{\partial T}\left[\cosh(\beta J)\right]$$

$$= k_B \ln(2\cosh\beta J) + \frac{k_B T}{\cosh(\beta J)} \sinh(\beta J) J \frac{d\beta}{dT}$$

$$= k_B \ln(2\cosh\beta J) - \frac{J}{T}\tanh\beta J$$

$$s = k_B\left[\ln(2\cosh\beta J) - \beta J \tanh\beta J\right]$$

At $T\to\infty$, $\beta\to 0$,

$$\cosh\beta J \simeq 1+\tfrac{1}{2}(\beta J)^2$$

$$\tanh(\beta J) \simeq \beta J$$

$$s \simeq k_B\left[\ln\left[2+(\beta J)^2\right] - (\beta J)^2\right]$$

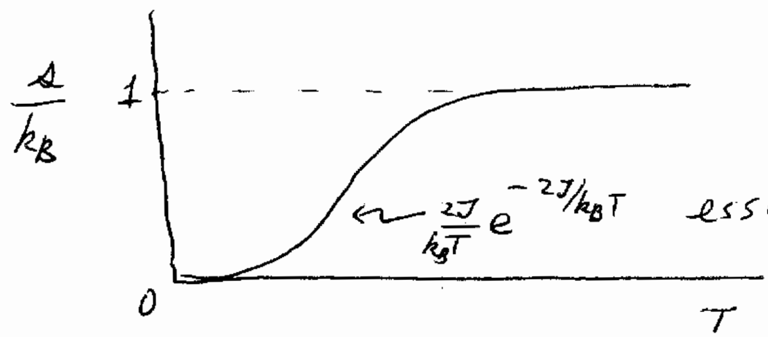
$$\simeq k_B \ln 2$$

At $T\to 0$, $\beta\to\infty$

$$\cosh\beta J \simeq e^{\beta J}$$

$$\tanh \simeq \frac{1-e^{-2\beta J}}{1+e^{-2\beta J}} \simeq 1-2e^{-2\beta J}$$

$$s \simeq k_B\left[\ln e^{\beta J} - \beta J(1-2e^{-2\beta J})\right] \simeq 2\frac{J}{T}e^{-2J/k_B T}$$



essential singularity at $T=0$

$$C = T\left(\frac{\partial s}{\partial T}\right) = k_B T\left\{ \cancel{\frac{-2J\sinh\beta J}{2\cosh\beta J}\frac{1}{k_BT^2}} + \cancel{\frac{J}{k_BT^2}\tanh\beta J} + \frac{\beta J^2}{k_BT^2}\frac{\partial}{\partial(\beta J)}\tanh\beta J \right\}$$

$$= \frac{J^2}{k_BT^2}\frac{\partial}{\partial(\beta J)}\left(\tanh\beta J\right) = \frac{J^2}{k_BT^2}\frac{1}{(\cosh\beta J)^2}$$

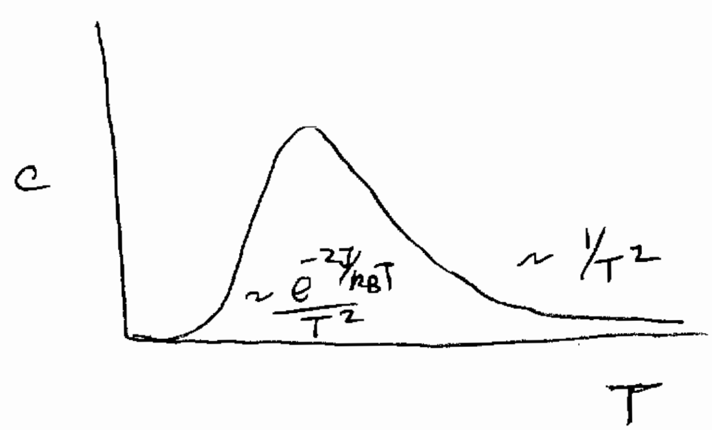
$$\boxed{C = k_B\left(\frac{\beta J}{\cosh\beta J}\right)^2}$$

as $T\to\infty$, $\beta\to 0$

$$C \simeq k_B\left(\frac{J}{k_BT}\right)^2$$

as $T\to 0$, $\beta\to\infty$

$$C \simeq k_B\left(\frac{J}{k_BT}\right)^2 e^{-2J/k_BT}$$



essential singularity at $T=0$

$\Rightarrow$ No singularity at any finite T.

$\Rightarrow$ No phase transition at any finite T