**Discussion Question 5**

Distinguishable particles

The probability density $\rho^{\text{dis}}$ for the system of distinguishable particles to have the $N$ particles at coordinates $\{x_i\}$ with momena $\{p_i\}$ is,

$$\rho^{\text{dis}}(\{x_i, p_i\}) = \frac{\mathrm{e}^{-\beta\mathcal{H}(\{x_i,p_i\})}}{\left(\prod_i \int dx_i dp_i\right) \mathrm{e}^{-\beta\mathcal{H}(\{x_i,p_i\})}} \tag{1}$$

where $\rho^{\text{dis}}$ is normalized so that

$$\int dx_1 dp_1 \cdots dx_N dp_N \, \rho^{\text{dis}}(x_1, p_1, \ldots, x_N, p_N) = 1 \tag{2}$$

Since the particles are non-interacting, $\mathcal{H}(\{x_i, p_i\}) = \sum_{i=1}^{N} \mathcal{H}^{(1)}(x_i, p_i)$, and this becomes,

$$\rho^{\text{dis}}(\{x_i, p_i\}) = \frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)} \cdots \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}{\left(\int dx_1 dp_1 \, \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}\right) \cdots \left(\int dx_N dp_N \, \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}\right)} \tag{3}$$

$$= \left(\frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}}{\int dx_1 dp_1 \, \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}}\right) \cdots \left(\frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}{\int dx_N dp_N \, \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}\right) \tag{4}$$

$$= \rho_1(x_1, p_1) \cdots \rho_1(x_N, p_N) \qquad \text{where} \qquad \rho_1(x, p) \equiv \frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x,p)}}{\int dx dp \, \mathrm{e}^{-\beta\mathcal{H}^{(1)}(x,p)}} \tag{5}$$

Since the particles are non-interacting, they are statistically independent, so the joint $N$-particle probability density $\rho^{\text{dis}}(\{x_i, p_i\})$ factors into a product of $N$ single-particle probability densities $\rho_1(x, p)$. That is always true of independent random variables – the joint probability distribution factors into a product of distributions for the individual random variables.

Now we are interested only in the probability for the position, so we integrate over the momentum. Since $\mathcal{H}^{(1)} = \frac{p^2}{2m} + U(x)$ we have

$$\rho_1(x) = \int dp \, \rho_1(x, p) = \frac{\mathrm{e}^{-\beta U(x)} \int dp \, \mathrm{e}^{-\beta p^2/2m}}{\int dx \, \mathrm{e}^{-\beta U(x)} \int dp \, \mathrm{e}^{-\beta p^2/2m}} = \frac{\mathrm{e}^{-\beta U(x)}}{\int dx \, \mathrm{e}^{-\beta U(x)}} \tag{6}$$

For $U(x) = \begin{cases} 0 & 0 \le x < L/2 \\ U_0 & L/2 \le x \le L \end{cases}$ we have $\int dx \, \mathrm{e}^{-\beta U(x)} = \frac{L}{2}\left[1 + \mathrm{e}^{\beta U_0}\right]$, so

$$\boxed{\rho_1(x) = \frac{2\,\mathrm{e}^{-\beta U(x)}}{L\left[1 + \mathrm{e}^{-\beta U_0}\right]}} \tag{7}$$

The probability the particle will be found in the right hand side of the box is then,

$$p = \int_{L/2}^{L} dx\, \rho_1(x) = \frac{L}{2}\, \frac{2\mathrm{e}^{-\beta U_0}}{L\,[1+\mathrm{e}^{-\beta U_0}]} = \boxed{\frac{\mathrm{e}^{-\beta U_0}}{[1+\mathrm{e}^{-\beta U_0}]} = p} \tag{8}$$

and the probability the particle will be found in the left hand side of the box is,

$$\boxed{q = 1 - p = \frac{1}{[1+\mathrm{e}^{-\beta U_0}]}} \tag{9}$$

Back now to the $N$-particle system, the probability that we have particles $i$ at positions $x_i$ is given by,

$$\rho^{\mathrm{dis}}(x_1,\ldots,x_N) = \rho_1(x_1)\cdots\rho_1(x_N) \qquad \text{since we just integrate Eq. (5) over all the } p_i \tag{10}$$

The probability that we will have the specific particles $i = 1,\ldots,M$ on the right side, and $i = M+1,\ldots,N$ on the left side, is then obtained by integrating each of the $\rho_1(x)$ over the appropriate interval. We get,

$$P = p^M q^{N-M} \tag{11}$$

But if we want to know the probability that $M$ of the particles are on the right side, and all the others are on the left side, and we don't care which are the ones that are on the right, then that probability is,

$$\boxed{P(M) = \frac{N!}{M!(N-M)!}\, p^M q^{N-M}} \tag{12}$$

since there are $\dfrac{N!}{M!(N-M)!}$ ways to choose which $M$ of the $N$ particles to put on the right side.

Indistinguishable particles

Now suppose our particles are non-interacting but are indistinguishable. Now the $N$-particle probability density $\rho^{\mathrm{indis}}$ should be normalized so,

$$\frac{1}{N!}\int dx_1 dp_1 \cdots dx_N dp_N\, \rho^{\mathrm{indis}}(x_1,p_1,\ldots,x_N,p_N) = 1 \tag{13}$$

The $1/N!$ is there because we do not want to over-count states, i.e. the configuration $(x_1,p_1,x_2,p_2,\ldots,x_N,p_N)$ is the same as the configuration $(x_2,p_2,x_1,p_1,\ldots,x_N,p_N)$. So $\rho(x_1,p_1,\ldots,x_N,p_N)$ is the probability density that one particle has coordinates $(x_1,p_1)$, another has coordinates $(x_2,p_2)$, and so on, and we don't care which particle has which coordinates because they are indistinguishable.

Comparing to Eq. (2) we can therefore write,

$$\rho^{\mathrm{indis}}(\{x_i,p_i\}) = N!\,\rho^{\mathrm{dis}}(\{x_i,p_i\}) \tag{14}$$

And similarly, integrating over the momenta, the joint probability to find one particle at $x_1$, another at $x_2$, and so on, is,

$$\rho^{\mathrm{indis}}(x_1,x_2,\ldots,x_N) = N!\,\rho^{\mathrm{dis}}(x_1,x_2,\ldots,x_N) = N!\,\rho_1(x_1)\cdots\rho_1(x_N) \tag{15}$$

Now suppose I have $M$ red particles and $N-M$ blue particles in the box. The red particles are indistinguishable from each other, and the blue particles are indistinguishable from each other, but the red particles can be distinguished from the blue particles. The probability that the red particles are at $(x_1, \ldots, x_M)$ and the blue particles are at $(x_{M+1}, x_N)$ would be,

$$\rho^{\text{indis}}(x_1, \ldots, x_M)\, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N) = \Big[M!\, \rho(x_1) \cdots \rho(x_M)\Big] \Big[(N-M)!\, \rho(x_{M+1}) \cdots \rho(x_N)\Big] \tag{16}$$

$$= M!(N-M)!\, \rho_1(x_1) \cdots \rho_1(x_N) \tag{17}$$

Comparing to Eq. (15) we therefore have,

$$\boxed{\rho^{\text{indis}}(x_1, x_2, \ldots, x_N) = \frac{N!}{M!(N-M)!}\, \rho^{\text{indis}}(x_1, \ldots, x_M)\, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N)} \tag{18}$$

If we recall that in the *microcanonical* ensemble, the probability to be in a particular state is $1/\Omega$, then the above is similar to Eq. (2.7.25) in our discussion of the entropy of mixing.

So, using Eq. (18), the probability $P(M)$ that $M$ of the indistinguishable particles are on the right and $N-M$ are on the left is,

$$P(M) = \int\limits_{\substack{\text{such that} \\ M \text{ of the } x_i \text{ have } L/2 \le x_i \\ N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_1 \cdots dx_N\, \rho^{\text{indis}}(x_1, \cdots, x_N) \tag{19}$$

$$= \frac{N!}{M!(N-M)!} \int\limits_{\substack{\text{such that} \\ \text{all } M \text{ of the } x_i \text{ have } L/2 \le x_i \\ \text{without double counting configurations}}} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \cdots, x_M) \tag{20}$$

$$\times \int\limits_{\substack{\text{such that} \\ \text{all } N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_{M+1} \cdots dx_N\, \rho^{\text{indis}}(x_{M+1}, \cdots, x_N)$$

Now we have for the first term on the rightmost side of the above equation,

$$P_R = \int\limits_{\substack{\text{such that} \\ \text{all } M \text{ of the } x_i \text{ have } L/2 \le x_i \\ \text{without double counting configurations}}} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \cdots, x_M) \tag{21}$$

$$= \frac{1}{M!} \int_{L/2}^{L} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \ldots, x_M) = \frac{1}{M!} \int_{L/2}^{L} dx_1 \cdots dx_M \Big[M!\, \rho^{\text{dis}}(x_1, \ldots, x_M)\Big] \tag{22}$$

$$= \int_{L/2}^{L} dx_1 \cdots dx_M\, \rho_1(x_1) \cdots \rho_1(x_M) = p^M \tag{23}$$

while the second term is,

$$P_L = \int_{\substack{\text{such that} \\ \text{all } N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_{M+1} \cdots dx_N \, \rho^{\text{indis}}(x_{M+1}, \cdots, x_N) \tag{24}$$

$$= \frac{1}{(N-M)!} \int_0^{L/2} dx_{M+1} \cdots dx_N \, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N) \tag{25}$$

$$= \frac{1}{(N-M)!} \int_0^{L/2} dx_{M+1} \cdots dx_N \left[ (N-M)! \, \rho^{\text{dis}}(x_{M+1}, \ldots, x_N) \right] \tag{26}$$

$$= \int_0^{L/2} dx_{M+1} \cdots dx_N \, \rho_1(x_{M+1}) \ldots \rho_1(x_N) = q^{N-M} \tag{27}$$

Putting these results into Eq. (20) we get,

$$P(M) = \frac{N!}{M!(N-M)!} P_R \, P_L = \frac{N!}{M!(N-M)!} p^N \, q^{N-M} \tag{28}$$

This is exactly the same answer that we had for distinguishable particles!

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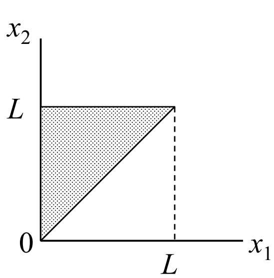
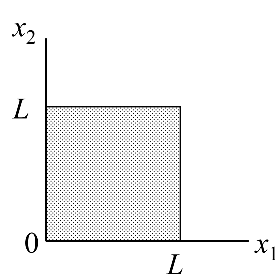
In several places above we discussed doing integrals without double counting states for identical particles. To be specific about what we mean, suppose the coordinates of the $N$ particles are $(x_1, p_1)$, $(x_2, p_2)$, $\ldots (x_N, p_N)$. Then if we want to integrate without double counting, we should integrate the normalization condition as,

$$\int_{-\infty}^{\infty} dp_N \cdots \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_1 \int_{x_{N-1}}^{L} dx_N \cdots \int_{x_1}^{L} dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, p_1, x_2, p_2, \ldots, x_N, p_N) = 1 \tag{29}$$

That is, we first choose $x_1 \in [0, L]$, then we should next choose $x_2 \in [x_1, L]$, then $x_3 \in [x_2, L]$, etc., so that the position coordinates are ordered as $0 \le x_1 \le x_2 \le \cdots \le x_N \le L$. This way if $(x_1, x_2)$ is in the region of integration, then $(x_2, x_1)$ is not, and so we do not double count. Alternatively, we could integrate over $x_i \in [0, L]$ for all $x_i$, but then we need to divide the integration by the factor $N!$ because we are double counting.

To see this graphically, consider the case of just two particles. By the above, we want to integrate over $x_1 \in [0, L]$ and $x_2 \in [x_1, L]$. Graphically this is the shaded region shown below to the left. Alternatively, we could integrate over $x_1 \in [0, L]$ *and* $x_2 \in [0, L]$, shown as the shaded region below to the right. But this region has twice the area as the one to the left, so we would have to multiply by $1/2 = 1/2!$ to get the same answer as when we integrate over the region to the left.




If we had distinguishable particles, then $(x_1, x_2)$ is a different state from $(x_2, x_1)$ and we would integrate over the region above to the right.

One then has (imagine we have already integrated over the $p_i$),

$$\frac{1}{2!} \int_0^L dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, x_2) = \int_{x_1}^{L} dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, x_2) = 1 \tag{30}$$

while

$$\int_0^L dx_2 \int_0^L dx_1\, \rho^{\mathrm{dis}}(x_1, x_2) = 1 \tag{31}$$

This leads to

$$\frac{1}{2!}\rho^{\mathrm{indis}}(x_1, x_2) = \rho^{\mathrm{dis}}(x_1, x_2) \quad \Rightarrow \quad \rho^{\mathrm{indis}}(x_1, x_2) = 2!\, \rho^{\mathrm{dis}}(x_1, x_2) \tag{32}$$

$\rho^{\mathrm{indis}}$ must be twice as large as $\rho^{\mathrm{dis}}$ because when we normalize we are really integrating $\rho^{\mathrm{indis}}$ over only halve the area as when we integrate $\rho^{\mathrm{dis}}$.

For $N$ particles, this generalizes to $\rho^{\mathrm{indis}}(x_1, \ldots, x_N) = N!\rho^{\mathrm{dis}}(x_1, \ldots, x_N)$.